APPENDIXES

A Numbers, Inequalities, and Absolute Values

1. $|5-23| = |-18| = 18$

2. $|5| - |-23| = 5 - 23 = -18$

3. $|-\pi| = \pi$ because $\pi > 0$.

4. $|\pi - 2| = \pi - 2$ because $\pi - 2 > 0$.

5. $\left|\sqrt{5}-5\right| = -\left(\sqrt{5}-5\right) = 5-\sqrt{5}$ because $\sqrt{5}-5 < 0$.

6. $\Big|\,|-2| - |-3|\Big| = |2-3| = |-1| = 1$

7. If $x < 2$, $x - 2 < 0$, so $|x-2| = -(x-2) = 2-x$.

8. If $x > 2$, $x - 2 > 0$, so $|x-2| = x-2$.

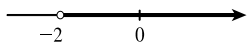
9. $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 < 0 \end{cases} = \begin{cases} x+1 & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1 \end{cases}$

10. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$

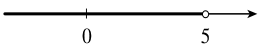
11. $\left|x^2+1\right| = x^2+1$ [since $x^2+1 \geq 0$ for all x].

12. Determine when $1-2x^2 < 0 \;\Leftrightarrow\; 1 < 2x^2 \;\Leftrightarrow\; x^2 > \frac{1}{2} \;\Leftrightarrow\; \sqrt{x^2} > \sqrt{\frac{1}{2}} \;\Leftrightarrow\; |x| > \sqrt{\frac{1}{2}} \;\Leftrightarrow$ $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$. Thus, $\left|1-2x^2\right| = \begin{cases} 1-2x^2 & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ 2x^2-1 & \text{if } x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}} \end{cases}$

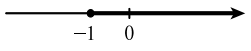
13. $2x+7 > 3 \;\Leftrightarrow\; 2x > -4 \;\Leftrightarrow\; x > -2$, so $x \in (-2, \infty)$.



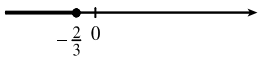
14. $3x - 11 < 4 \;\Leftrightarrow\; 3x < 15 \;\Leftrightarrow\; x < 5$, so $x \in (-\infty, 5)$.



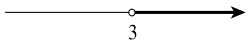
15. $1 - x \leq 2 \;\Leftrightarrow\; -x \leq 1 \;\Leftrightarrow\; x \geq -1$, so $x \in [-1, \infty)$.



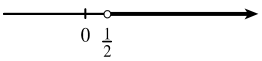
16. $4 - 3x \geq 6 \;\Leftrightarrow\; -3x \geq 2 \;\Leftrightarrow\; x \leq -\frac{2}{3}$, so $x \in \left(-\infty, -\frac{2}{3}\right]$.



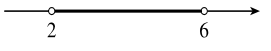
17. $2x + 1 < 5x - 8 \;\Leftrightarrow\; 9 < 3x \;\Leftrightarrow\; 3 < x$, so $x \in (3, \infty)$.



18. $1 + 5x > 5 - 3x \;\Leftrightarrow\; 8x > 4 \;\Leftrightarrow\; x > \frac{1}{2}$, so $x \in \left(\frac{1}{2}, \infty\right)$.



19. $-1 < 2x - 5 < 7 \;\Leftrightarrow\; 4 < 2x < 12 \;\Leftrightarrow\; 2 < x < 6$, so $x \in (2, 6)$.



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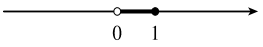
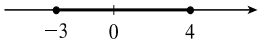
20. $1 < 3x + 4 \le 16 \quad \Leftrightarrow \quad -3 < 3x \le 12 \quad \Leftrightarrow \quad -1 < x \le 4$, so $x \in (-1, 4]$.

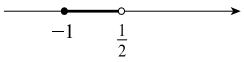
21. $0 \le 1 - x < 1 \quad \Leftrightarrow \quad -1 \le -x < 0 \quad \Leftrightarrow \quad 1 \ge x > 0$, so $x \in (0, 1]$.



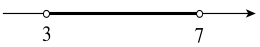
22. $-5 \le 3 - 2x \le 9 \quad \Leftrightarrow \quad -8 \le -2x \le 6 \quad \Leftrightarrow \quad 4 \ge x \ge -3$, so $x \in [-3, 4]$.



23. $4x < 2x + 1 \le 3x + 2$. So $4x < 2x + 1 \quad \Leftrightarrow \quad 2x < 1 \quad \Leftrightarrow \quad x < \frac{1}{2}$, and $2x + 1 \le 3x + 2 \quad \Leftrightarrow \quad -1 \le x$. Thus, $x \in \left[-1, \frac{1}{2}\right)$.



24. $2x - 3 < x + 4 < 3x - 2$. So $2x - 3 < x + 4 \quad \Leftrightarrow \quad x < 7$, and $x + 4 < 3x - 2 \quad \Leftrightarrow \quad 6 < 2x \quad \Leftrightarrow \quad 3 < x$, so $x \in (3, 7)$.

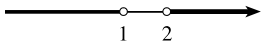


25. $(x - 1)(x - 2) > 0$.

Case 1: (both factors are positive, so their product is positive) $x - 1 > 0 \quad \Leftrightarrow \quad x > 1$, and $x - 2 > 0 \quad \Leftrightarrow \quad x > 2$, so $x \in (2, \infty)$.

Case 2: (both factors are negative, so their product is positive) $x - 1 < 0 \quad \Leftrightarrow \quad x < 1$, and $x - 2 < 0 \quad \Leftrightarrow \quad x < 2$, so $x \in (-\infty, 1)$.

Thus, the solution set is $(-\infty, 1) \cup (2, \infty)$.

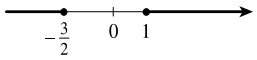


26. $(2x + 3)(x - 1) \ge 0$.

Case 1: $2x + 3 \ge 0 \quad \Leftrightarrow \quad x \ge -\frac{3}{2}$, and $x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge 1$, so $x \in [1, \infty)$.

Case 2: $2x + 3 \le 0 \quad \Leftrightarrow \quad x \le -\frac{3}{2}$, and $x - 1 \le 0 \quad \Leftrightarrow \quad x \le 1$, so $x \in \left(-\infty, -\frac{3}{2}\right]$.

Thus, the solution set is $\left(-\infty, -\frac{3}{2}\right] \cup [1, \infty)$.

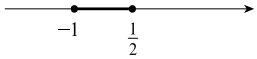


27. $2x^2 + x \le 1 \quad \Leftrightarrow \quad 2x^2 + x - 1 \le 0 \quad \Leftrightarrow \quad (2x - 1)(x + 1) \le 0$.

Case 1: $2x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge \frac{1}{2}$, and $x + 1 \le 0 \quad \Leftrightarrow \quad x \le -1$, which is an impossible combination.

Case 2: $2x - 1 \le 0 \quad \Leftrightarrow \quad x \le \frac{1}{2}$, and $x + 1 \ge 0 \quad \Leftrightarrow \quad x \ge -1$, so $x \in \left[-1, \frac{1}{2}\right]$.

Thus, the solution set is $\left[-1, \frac{1}{2}\right]$.

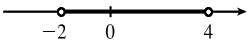


28. $x^2 < 2x + 8 \quad \Leftrightarrow \quad x^2 - 2x - 8 < 0 \quad \Leftrightarrow \quad (x - 4)(x + 2) < 0$.

Case 1: $x > 4$ and $x < -2$, which is impossible.

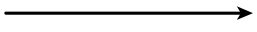
Case 2: $x < 4$ and $x > -2$.

Thus, the solution set is $(-2, 4)$.



29. $x^2 + x + 1 > 0 \quad \Leftrightarrow \quad x^2 + x + \frac{1}{4} + \frac{3}{4} > 0 \quad \Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. But since $\left(x + \frac{1}{2}\right)^2 \ge 0$ for every real x, the original inequality will be true for all real x as well.

Thus, the solution set is $(-\infty, \infty)$.



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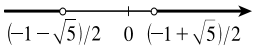
30. $x^2 + x > 1 \quad \Leftrightarrow \quad x^2 + x - 1 > 0$. Using the quadratic formula, we obtain

$x^2 + x - 1 = \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0.$

Case 1: $x - \frac{-1-\sqrt{5}}{2} > 0$ and $x - \frac{-1+\sqrt{5}}{2} > 0$, so that $x > \frac{-1+\sqrt{5}}{2}$.

Case 2: $x - \frac{-1-\sqrt{5}}{2} < 0$ and $x - \frac{-1+\sqrt{5}}{2} < 0$, so that $x < \frac{-1-\sqrt{5}}{2}$.

Thus, the solution set is $\left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \infty\right)$.

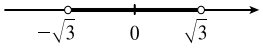


31. $x^2 < 3 \quad \Leftrightarrow \quad x^2 - 3 < 0 \quad \Leftrightarrow \quad \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) < 0.$

Case 1: $x > \sqrt{3}$ and $x < -\sqrt{3}$, which is impossible.

Case 2: $x < \sqrt{3}$ and $x > -\sqrt{3}$.

Thus, the solution set is $\left(-\sqrt{3}, \sqrt{3}\right)$.



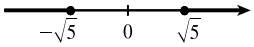
Another method: $x^2 < 3 \quad \Leftrightarrow \quad |x| < \sqrt{3} \quad \Leftrightarrow \quad -\sqrt{3} < x < \sqrt{3}$.

32. $x^2 \geq 5 \quad \Leftrightarrow \quad x^2 - 5 \geq 0 \quad \Leftrightarrow \quad \left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right) \geq 0.$

Case 1: $x \geq \sqrt{5}$ and $x \geq -\sqrt{5}$, so $x \in \left[\sqrt{5}, \infty\right)$.

Case 2: $x \leq \sqrt{5}$ and $x \leq -\sqrt{5}$, so $x \in \left(-\infty, -\sqrt{5}\right]$.

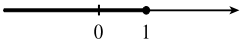
Thus, the solution set is $\left(-\infty, -\sqrt{5}\right] \cup \left[\sqrt{5}, \infty\right)$.



Another method: $x^2 \geq 5 \quad \Leftrightarrow \quad |x| \geq \sqrt{5} \quad \Leftrightarrow \quad x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.

33. $x^3 - x^2 \leq 0 \quad \Leftrightarrow \quad x^2(x-1) \leq 0$. Since $x^2 \geq 0$ for all x, the inequality is satisfied when $x - 1 \leq 0 \quad \Leftrightarrow \quad x \leq 1$.

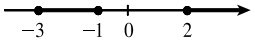
Thus, the solution set is $(-\infty, 1]$.



34. $(x+1)(x-2)(x+3) = 0 \quad \Leftrightarrow \quad x = -1, 2,$ or -3. Construct a chart:

Interval	$x+1$	$x-2$	$x+3$	$(x+1)(x-2)(x+3)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < -1$	$-$	$-$	$+$	$+$
$-1 < x < 2$	$+$	$-$	$+$	$-$
$x > 2$	$+$	$+$	$+$	$+$

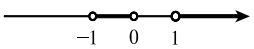
Thus, $(x+1)(x-2)(x+3) \geq 0$ on $[-3, -1]$ and $[2, \infty)$, and the solution set is $[-3, -1] \cup [2, \infty)$.



35. $x^3 > x \quad \Leftrightarrow \quad x^3 - x > 0 \quad \Leftrightarrow \quad x\left(x^2 - 1\right) > 0 \quad \Leftrightarrow \quad x(x-1)(x+1) > 0$. Construct a chart:

Interval	x	$x-1$	$x+1$	$x(x-1)(x+1)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

Since $x^3 > x$ when the last column is positive, the solution set is $(-1, 0) \cup (1, \infty)$.



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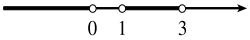
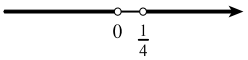
36. $x^3 + 3x < 4x^2 \quad \Leftrightarrow \quad x^3 - 4x^2 + 3x < 0 \quad \Leftrightarrow \quad x\left(x^2 - 4x + 3\right) < 0 \quad \Leftrightarrow \quad x(x-1)(x-3) < 0.$

Interval	x	$x-1$	$x-3$	$x(x-1)(x-3)$
$x < 0$	$-$	$-$	$-$	$-$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

Thus, the solution set is $(-\infty, 0) \cup (1, 3)$.

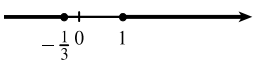


37. $1/x < 4$. This is clearly true for $x < 0$. So suppose $x > 0$. then $1/x < 4 \quad \Leftrightarrow$ $1 < 4x \quad \Leftrightarrow \quad \frac{1}{4} < x$. Thus, the solution set is $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$.



38. $-3 < 1/x \leq 1$. We solve the two inequalities separately and take the intersection of the solution sets. First, $-3 < 1/x$ is clearly true for $x > 0$. So suppose $x < 0$. Then $-3 < 1/x \quad \Leftrightarrow \quad -3x > 1 \quad \Leftrightarrow \quad x < -\frac{1}{3}$, so for this inequality, the solution set is $\left(-\infty, -\frac{1}{3}\right) \cup (0, \infty)$. Now $1/x \leq 1$ is clearly true if $x < 0$. So suppose $x > 0$. Then $1/x \leq 1 \quad \Leftrightarrow \quad 1 \leq x$, and the solution set here is $(-\infty, 0) \cup [1, \infty)$.

Taking the intersection of the two solution sets gives the final solution set: $\left(-\infty, -\frac{1}{3}\right) \cup [1, \infty)$.



39. $C = \frac{5}{9}(F - 32) \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. So $50 \leq F \leq 95 \quad \Rightarrow \quad 50 \leq \frac{9}{5}C + 32 \leq 95 \quad \Rightarrow \quad 18 \leq \frac{9}{5}C \leq 63 \quad \Rightarrow$ $10 \leq C \leq 35$. So the interval is $[10, 35]$.

40. Since $20 \leq C \leq 30$ and $C = \frac{5}{9}(F - 32)$, we have $20 \leq \frac{5}{9}(F - 32) \leq 30 \quad \Rightarrow \quad 36 \leq F - 32 \leq 54 \quad \Rightarrow \quad 68 \leq F \leq 86$. So the interval is $[68, 86]$.

41. (a) Let T represent the temperature in degrees Celsius and h the height in km. $T = 20$ when $h = 0$ and T decreases by $10°\text{C}$ for every km ($1°\text{C}$ for each 100-m rise). Thus, $T = 20 - 10h$ when $0 \leq h \leq 12$.

(b) From part (a), $T = 20 - 10h \quad \Rightarrow \quad 10h = 20 - T \quad \Rightarrow \quad h = 2 - T/10$. So $0 \leq h \leq 5 \quad \Rightarrow \quad 0 \leq 2 - T/10 \leq 5 \quad \Rightarrow$ $-2 \leq -T/10 \leq 3 \quad \Rightarrow \quad -20 \leq -T \leq 30 \quad \Rightarrow \quad 20 \geq T \geq -30 \quad \Rightarrow \quad -30 \leq T \leq 20$. Thus, the range of temperatures (in $°\text{C}$) to be expected is $[-30, 20]$.

42. The ball will be at least 32 ft above the ground if $h \geq 32 \quad \Leftrightarrow \quad 128 + 16t - 16t^2 \geq 32 \quad \Leftrightarrow \quad 16t^2 - 16t - 96 \leq 0 \quad \Leftrightarrow$ $16(t-3)(t+2) \leq 0$. $t = 3$ and $t = -2$ are endpoints of the interval we're looking for, and constructing a table gives $-2 \leq t \leq 3$. But $t \geq 0$, so the ball will be at least 32 ft above the ground in the time interval $[0, 3]$.

43. $|2x| = 3 \quad \Leftrightarrow \quad$ either $2x = 3$ or $2x = -3 \quad \Leftrightarrow \quad x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

44. $|3x + 5| = 1 \quad \Leftrightarrow \quad$ either $3x + 5 = 1$ or -1. In the first case, $3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$, and in the second case, $3x = -6 \quad \Leftrightarrow \quad x = -2$. So the solutions are -2 and $-\frac{4}{3}$.

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45. $|x+3| = |2x+1| \quad \Leftrightarrow \quad$ either $x+3 = 2x+1$ or $x+3 = -(2x+1)$. In the first case, $x = 2$, and in the second case, $x+3 = -2x-1 \quad \Leftrightarrow \quad 3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$. So the solutions are $-\frac{4}{3}$ and 2.

46. $\left|\dfrac{2x-1}{x+1}\right| = 3 \quad \Leftrightarrow \quad$ either $\dfrac{2x-1}{x+1} = 3$ or $\dfrac{2x-1}{x+1} = -3$. In the first case, $2x-1 = 3x+3 \quad \Leftrightarrow \quad x = -4$, and in the second case, $2x-1 = -3x-3 \quad \Leftrightarrow \quad x = -\frac{2}{5}$.

47. By Property 5 of absolute values, $|x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$, so $x \in (-3, 3)$.

48. By Properties 4 and 6 of absolute values, $|x| \geq 3 \quad \Leftrightarrow \quad x \leq -3$ or $x \geq 3$, so $x \in (-\infty, -3] \cup [3, \infty)$.

49. $|x-4| < 1 \quad \Leftrightarrow \quad -1 < x-4 < 1 \quad \Leftrightarrow \quad 3 < x < 5$, so $x \in (3, 5)$.

50. $|x-6| < 0.1 \quad \Leftrightarrow \quad -0.1 < x-6 < 0.1 \quad \Leftrightarrow \quad 5.9 < x < 6.1$, so $x \in (5.9, 6.1)$.

51. $|x+5| \geq 2 \quad \Leftrightarrow \quad x+5 \geq 2$ or $x+5 \leq -2 \quad \Leftrightarrow \quad x \geq -3$ or $x \leq -7$, so $x \in (-\infty, -7] \cup [-3, \infty)$.

52. $|x+1| \geq 3 \quad \Leftrightarrow \quad x+1 \geq 3$ or $x+1 \leq -3 \quad \Leftrightarrow \quad x \geq 2$ or $x \leq -4$, so $x \in (-\infty, -4] \cup [2, \infty)$.

53. $|2x-3| \leq 0.4 \quad \Leftrightarrow \quad -0.4 \leq 2x-3 \leq 0.4 \quad \Leftrightarrow \quad 2.6 \leq 2x \leq 3.4 \quad \Leftrightarrow \quad 1.3 \leq x \leq 1.7$, so $x \in [1.3, 1.7]$.

54. $|5x-2| < 6 \quad \Leftrightarrow \quad -6 < 5x-2 < 6 \quad \Leftrightarrow \quad -4 < 5x < 8 \quad \Leftrightarrow \quad -\frac{4}{5} < x < \frac{8}{5}$, so $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$.

55. $1 \leq |x| \leq 4$. So either $1 \leq x \leq 4$ or $1 \leq -x \leq 4 \quad \Leftrightarrow \quad -1 \geq x \geq -4$. Thus, $x \in [-4, -1] \cup [1, 4]$.

56. $0 < |x-5| < \frac{1}{2}$. Clearly $0 < |x-5|$ for $x \neq 5$. Now $|x-5| < \frac{1}{2} \quad \Leftrightarrow \quad -\frac{1}{2} < x-5 < \frac{1}{2} \quad \Leftrightarrow \quad 4.5 < x < 5.5$. So the solution set is $(4.5, 5) \cup (5, 5.5)$.

57. $a(bx-c) \geq bc \quad \Leftrightarrow \quad bx-c \geq \dfrac{bc}{a} \quad \Leftrightarrow \quad bx \geq \dfrac{bc}{a} + c = \dfrac{bc+ac}{a} \quad \Leftrightarrow \quad x \geq \dfrac{bc+ac}{ab}$

58. $a \leq bx+c < 2a \quad \Leftrightarrow \quad a-c \leq bx < 2a-c \quad \Leftrightarrow \quad \dfrac{a-c}{b} \leq x < \dfrac{2a-c}{b}$ (since $b > 0$)

59. $ax+b < c \quad \Leftrightarrow \quad ax < c-b \quad \Leftrightarrow \quad x > \dfrac{c-b}{a}$ [since $a < 0$]

60. $\dfrac{ax+b}{c} \leq b \quad \Leftrightarrow \quad ax+b \geq bc$ [since $c < 0$] $\quad \Leftrightarrow \quad ax \geq bc-b \quad \Leftrightarrow \quad x \leq \dfrac{b(c-1)}{a}$ [since $a < 0$]

61. $|(x+y)-5| = |(x-2)+(y-3)| \leq |x-2| + |y-3| < 0.01 + 0.04 = 0.05$

62. Use the Triangle Inequality: $|x+3| < \frac{1}{2} \quad \Rightarrow$

$|4x+13| = |4(x+3)+1| \leq |4(x+3)| + |1| = 4|x+3| + 1 < 4\left(\frac{1}{2}\right) + 1 = 3$

Another method: $|x+3| < \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2} < x+3 < \frac{1}{2} \quad \Rightarrow \quad -2 < 4x+12 < 2 \quad \Rightarrow \quad -1 < 4x+13 < 3 \quad \Rightarrow$ $|4x+13| < 3$

63. If $a < b$ then $a+a < a+b$ and $a+b < b+b$. So $2a < a+b < 2b$. Dividing by 2, we get $a < \frac{1}{2}(a+b) < b$.

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64. If $0 < a < b$, then $\dfrac{1}{ab} > 0$. So $a < b \quad \Rightarrow \quad \dfrac{1}{ab} \cdot a < \dfrac{1}{ab} \cdot b \quad \Leftrightarrow \quad \dfrac{1}{b} < \dfrac{1}{a}$.

65. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2 b^2} = \sqrt{a^2}\,\sqrt{b^2} = |a|\,|b|$

66. $\left|\dfrac{a}{b}\right| |b| = \left|\dfrac{a}{b} \cdot b\right| = |a|$ [using the result of Exercise 65]. Dividing the equation through by $|b|$ gives $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. If $0 < a < b$, then $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$ [using Rule 3 of Inequalities]. So $a^2 < ab < b^2$ and hence $a^2 < b^2$.

68. Following the hint, the Triangle Inequality becomes $|(x - y) + y| \le |x - y| + |y| \quad \Leftrightarrow \quad |x| \le |x - y| + |y| \quad \Leftrightarrow$ $|x - y| \ge |x| - |y|$.

69. Observe that the sum, difference and product of two integers is always an integer. Let the rational numbers be represented by $r = m/n$ and $s = p/q$ (where m, n, p and q are integers with $n \neq 0$, $q \neq 0$). Now $r + s = \dfrac{m}{n} + \dfrac{p}{q} = \dfrac{mq + pn}{nq}$, but $mq + pn$ and nq are both integers, so $\dfrac{mq + pn}{nq} = r + s$ is a rational number by definition. Similarly, $r - s = \dfrac{m}{n} - \dfrac{p}{q} = \dfrac{mq - pn}{nq}$ is a rational number. Finally, $r \cdot s = \dfrac{m}{n} \cdot \dfrac{p}{q} = \dfrac{mp}{nq}$ but mp and nq are both integers, so $\dfrac{mp}{nq} = r \cdot s$ is a rational number by definition.

70. (a) No. Consider the case of $\sqrt{2}$ and $-\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} + \left(-\sqrt{2}\right) = 0$ and 0, being an integer, is not irrational.

(b) No. Consider the case of $\sqrt{2}$ and $\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational.

B Coordinate Geometry and Lines

1. Use the distance formula with $P_1(x_1, y_1) = (1, 1)$ and $P_2(x_2, y_2) = (4, 5)$ to get

$$|P_1P_2| = \sqrt{(4-1)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

2. The distance from $(1, -3)$ to $(5, 7)$ is $\sqrt{(5-1)^2 + [7-(-3)]^2} = \sqrt{4^2 + 10^2} = \sqrt{116} = 2\sqrt{29}$.

3. The distance from $(6, -2)$ to $(-1, 3)$ is $\sqrt{-1-6)^2 + [3-(-2)]^2} = \sqrt{(-7)^2 + 5^2} = \sqrt{74}$.

4. The distance from $(1, -6)$ to $(-1, -3)$ is $\sqrt{(-1-1)^2 + [-3-(-6)]^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$.

5. The distance from $(2, 5)$ to $(4, -7)$ is $\sqrt{(4-2)^2 + (-7-5)^2} = \sqrt{2^2 + (-12)^2} = \sqrt{148} = 2\sqrt{37}$.

6. The distance from (a, b) to (b, a) is $\sqrt{(b-a)^2 + (a-b)^2} = \sqrt{(a-b)^2 + (a-b)^2} = \sqrt{2(a-b)^2} = \sqrt{2}\,|a-b|$.

7. The slope m of the line through $P(1, 5)$ and $Q(4, 11)$ is $m = \dfrac{11-5}{4-1} = \dfrac{6}{3} = 2$.

8. The slope m of the line through $P(-1, 6)$ and $Q(4, -3)$ is $m = \dfrac{-3-6}{4-(-1)} = -\dfrac{9}{5}$.

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9. The slope m of the line through $P(-3,3)$ and $Q(-1,-6)$ is $m = \dfrac{-6-3}{-1-(-3)} = -\dfrac{9}{2}$.

10. The slope m of the line through $P(-1,-4)$ and $Q(6,0)$ is $m = \dfrac{0-(-4)}{6-(-1)} = \dfrac{4}{7}$.

11. Using $A(0,2)$, $B(-3,-1)$, and $C(-4,3)$, we have $|AC| = \sqrt{(-4-0)^2+(3-2)^2} = \sqrt{(-4)^2+1^2} = \sqrt{17}$ and $|BC| = \sqrt{[-4-(-3)]^2+[3-(-1)]^2} = \sqrt{(-1)^2+4^2} = \sqrt{17}$, so the triangle has two sides of equal length, and is isosceles.

12. (a) Using $A(6,-7)$, $B(11,-3)$, and $C(2,-2)$, we have

$|AB| = \sqrt{(11-6)^2+[-3-(-7)]^2} = \sqrt{5^2+4^2} = \sqrt{41}$,

$|AC| = \sqrt{(2-6)^2+[-2-(-7)]^2} = \sqrt{(-4)^2+5^2} = \sqrt{41}$, and

$|BC| = \sqrt{(2-11)^2+[-2-(-3)]^2} = \sqrt{(-9)^2+1^2} = \sqrt{82}$.

Thus, $|AB|^2+|AC|^2 = 41+41 = 82 = |BC|^2$ and so $\triangle ABC$ is a right triangle.

(b) $m_{AB} = \dfrac{-3-(-7)}{11-6} = \dfrac{4}{5}$ and $m_{AC} = \dfrac{-2-(-7)}{2-6} = -\dfrac{5}{4}$. Thus $m_{AB}\cdot m_{AC} = -1$ and so AB is perpendicular to AC and $\triangle ABC$ must be a right triangle.

(c) Taking lengths from part (a), the base is $\sqrt{41}$ and the height is $\sqrt{41}$. Thus the area is $\frac{1}{2}bh = \frac{1}{2}\sqrt{41}\sqrt{41} = \frac{41}{2}$.

13. Using $A(-2,9)$, $B(4,6)$, $C(1,0)$, and $D(-5,3)$, we have

$|AB| = \sqrt{[4-(-2)]^2+(6-9)^2} = \sqrt{6^2+(-3)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|BC| = \sqrt{(1-4)^2+(0-6)^2} = \sqrt{(-3)^2+(-6)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|CD| = \sqrt{(-5-1)^2+(3-0)^2} = \sqrt{(-6)^2+3^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$, and

$|DA| = \sqrt{[-2-(-5)]^2+(9-3)^2} = \sqrt{3^2+6^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$. So all sides are of equal length and we have a rhombus. Moreover, $m_{AB} = \dfrac{6-9}{4-(-2)} = -\dfrac{1}{2}$, $m_{BC} = \dfrac{0-6}{1-4} = 2$, $m_{CD} = \dfrac{3-0}{-5-1} = -\dfrac{1}{2}$, and

$m_{DA} = \dfrac{9-3}{-2-(-5)} = 2$, so the sides are perpendicular. Thus, A, B, C, and D are vertices of a square.

14. (a) Using $A(-1,3)$, $B(3,11)$, and $C(5,15)$, we have

$|AB| = \sqrt{[3-(-1)]^2+(11-3)^2} = \sqrt{4^2+8^2} = \sqrt{80} = 4\sqrt{5}$,

$|BC| = \sqrt{(5-3)^2+(15-11)^2} = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$, and

$|AC| = \sqrt{[5-(-1)]^2+(15-3)^2} = \sqrt{6^2+12^2} = \sqrt{180} = 6\sqrt{5}$. Thus, $|AC| = |AB|+|BC|$.

(b) $m_{AB} = \dfrac{11-3}{3-(-1)} = \dfrac{8}{4} = 2$ and $m_{AC} = \dfrac{15-3}{5-(-1)} = \dfrac{12}{6} = 2$. Since the segments AB and AC have the same slope, A, B and C must be collinear.

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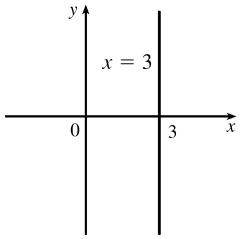
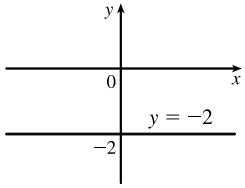
15. For the vertices $A(1,1)$, $B(7,4)$, $C(5,10)$, and $D(-1,7)$, the slope of the line segment AB is $\dfrac{4-1}{7-1}=\dfrac{1}{2}$, the slope of CD is $\dfrac{7-10}{-1-5}=\dfrac{1}{2}$, the slope of BC is $\dfrac{10-4}{5-7}=-3$, and the slope of DA is $\dfrac{1-7}{1-(-1)}=-3$. So AB is parallel to CD and BC is parallel to DA. Hence $ABCD$ is a parallelogram.

16. For the vertices $A(1,1)$, $B(11,3)$, $C(10,8)$, and $D(0,6)$, the slopes of the four sides are $m_{AB}=\dfrac{3-1}{11-1}=\dfrac{1}{5}$, $m_{BC}=\dfrac{8-3}{10-11}=-5$, $m_{CD}=\dfrac{6-8}{0-10}=\dfrac{1}{5}$, and $m_{DA}=\dfrac{1-6}{1-0}=-5$. Hence $AB \parallel CD$, $BC \parallel DA$, $AB \perp BC$, $BC \perp CD$, $CD \perp DA$, and $DA \perp AB$, and so $ABCD$ is a rectangle.

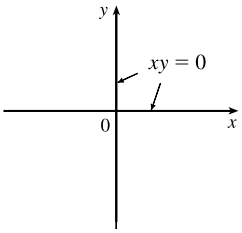
17. The graph of the equation $x=3$ is a vertical line with x-intercept 3. The line does not have a slope.



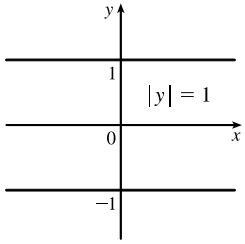
18. The graph of the equation $y=-2$ is a horizontal line with y-intercept -2. The line has slope 0.



19. $xy=0 \quad \Leftrightarrow \quad x=0$ or $y=0$. The graph consists of the coordinate axes.



20. $|y|=1 \quad \Leftrightarrow \quad y=1$ or $y=-1$



21. By the point-slope form of the equation of a line, an equation of the line through $(2,-3)$ with slope 6 is $y-(-3)=6(x-2)$ or $y=6x-15$.

22. $y-4=-3[x-(-1)]$ or $y=-3x+1$

23. $y-7=\frac{2}{3}(x-1)$ or $y=\frac{2}{3}x+\frac{19}{3}$

24. $y-(-5)=-\frac{7}{2}[x-(-3)]$ or $y=-\frac{7}{2}x-\frac{31}{2}$

25. The slope of the line through $(2,1)$ and $(1,6)$ is $m=\dfrac{6-1}{1-2}=-5$, so an equation of the line is $y-1=-5(x-2)$ or $y=-5x+11$.

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26. For $(-1,-2)$ and $(4,3)$, $m=\dfrac{3-(-2)}{4-(-1)}=1$. An equation of the line is $y-3=1(x-4)$ or $y=x-1$.

27. By the slope-intercept form of the equation of a line, an equation of the line is $y=3x-2$.

28. By the slope-intercept form of the equation of a line, an equation of the line is $y=\frac{2}{5}x+4$.

29. Since the line passes through $(1,0)$ and $(0,-3)$, its slope is $m=\dfrac{-3-0}{0-1}=3$, so an equation is $y=3x-3$.

Another method: From Exercise 61, $\dfrac{x}{1}+\dfrac{y}{-3}=1 \quad\Rightarrow\quad -3x+y=-3 \quad\Rightarrow\quad y=3x-3$.

30. For $(-8,0)$ and $(0,6)$, $m=\dfrac{6-0}{0-(-8)}=\dfrac{3}{4}$. So an equation is $y=\frac{3}{4}x+6$.

Another method: From Exercise 61, $\dfrac{x}{-8}+\dfrac{y}{6}=1 \quad\Rightarrow\quad -3x+4y=24 \quad\Rightarrow\quad y=\frac{3}{4}x+6$.

31. The line is parallel to the x-axis, so it is horizontal and must have the form $y=k$. Since it goes through the point $(x,y)=(4,5)$, the equation is $y=5$.

32. The line is parallel to the y-axis, so it is vertical and must have the form $x=k$. Since it goes through the point $(x,y)=(4,5)$, the equation is $x=4$.

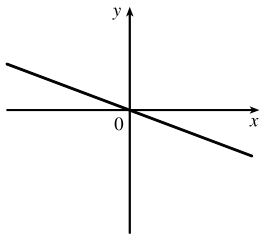
33. Putting the line $x+2y=6$ into its slope-intercept form gives us $y=-\frac{1}{2}x+3$, so we see that this line has slope $-\frac{1}{2}$. Thus, we want the line of slope $-\frac{1}{2}$ that passes through the point $(1,-6)$: $y-(-6)=-\frac{1}{2}(x-1) \quad\Leftrightarrow\quad y=-\frac{1}{2}x-\frac{11}{2}$.

34. $2x+3y+4=0 \quad\Leftrightarrow\quad y=-\frac{2}{3}x-\frac{4}{3}$, so $m=-\frac{2}{3}$ and the required line is $y=-\frac{2}{3}x+6$.

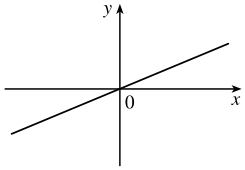
35. $2x+5y+8=0 \quad\Leftrightarrow\quad y=-\frac{2}{5}x-\frac{8}{5}$. Since this line has slope $-\frac{2}{5}$, a line perpendicular to it would have slope $\frac{5}{2}$, so the required line is $y-(-2)=\frac{5}{2}[x-(-1)] \quad\Leftrightarrow\quad y=\frac{5}{2}x+\frac{1}{2}$.

36. $4x-8y=1 \quad\Leftrightarrow\quad y=\frac{1}{2}x-\frac{1}{8}$. Since this line has slope $\frac{1}{2}$, a line perpendicular to it would have slope -2, so the required line is $y-\left(-\frac{2}{3}\right)=-2\left(x-\frac{1}{2}\right) \quad\Leftrightarrow\quad y=-2x+\frac{1}{3}$.

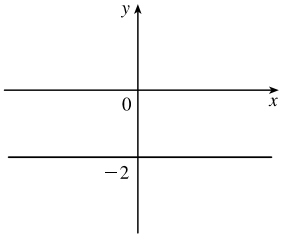
37. $x+3y=0 \quad\Leftrightarrow\quad y=-\frac{1}{3}x$, so the slope is $-\frac{1}{3}$ and the y-intercept is 0.



38. $2x-5y=0 \quad\Leftrightarrow\quad y=\frac{2}{5}x$, so the slope is $\frac{2}{5}$ and the y-intercept is 0.



39. $y=-2$ is a horizontal line with slope 0 and y-intercept -2.



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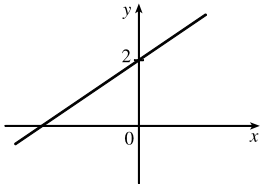
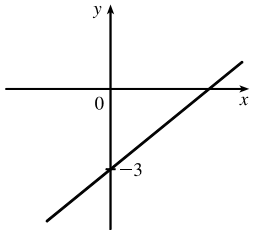
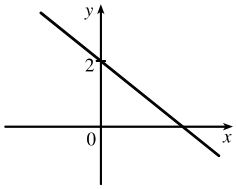
40. $2x - 3y + 6 = 0 \quad \Leftrightarrow$
$y = \frac{2}{3}x + 2$, so the slope is $\frac{2}{3}$ and the y-intercept is 2.



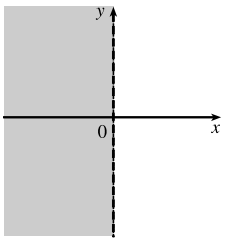
41. $3x - 4y = 12 \quad \Leftrightarrow$
$y = \frac{3}{4}x - 3$, so the slope is $\frac{3}{4}$ and the y-intercept is -3.



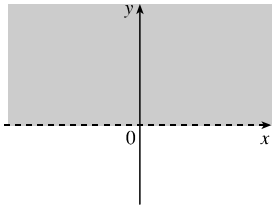
42. $4x + 5y = 10 \quad \Leftrightarrow$
$y = -\frac{4}{5}x + 2$, so the slope is $-\frac{4}{5}$ and the y-intercept is 2.



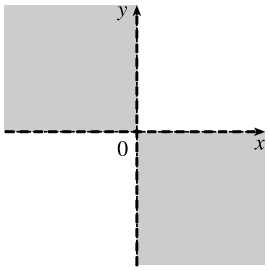
43. $\{(x, y) \mid x < 0\}$



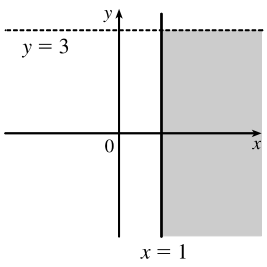
44. $\{(x, y) \mid y > 0\}$



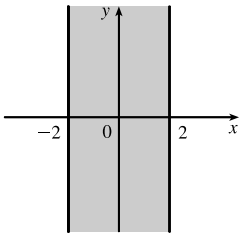
45. $\{(x, y) \mid xy < 0\} =$
$\{(x, y) \mid x < 0 \text{ and } y > 0\}$
$\cup \{(x, y) \mid x > 0 \text{ and } y < 0\}$



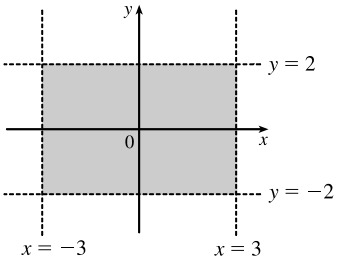
46. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$



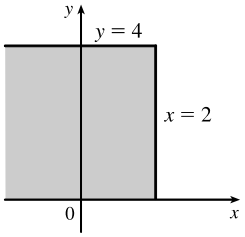
47. $\left\{(x, y) \,\middle|\, |x| \leq 2\right\} =$
$\{(x, y) \mid -2 \leq x \leq 2\}$



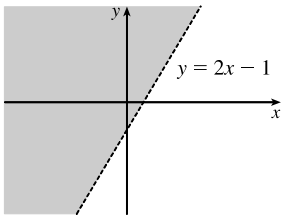
48. $\left\{(x, y) \,\middle|\, |x| < 3 \text{ and } |y| < 2\right\}$



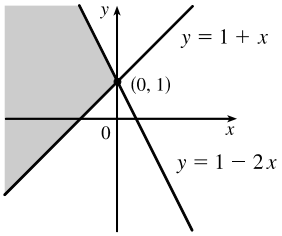
49. $\{(x, y) \mid 0 \leq y \leq 4, x \leq 2\}$



50. $\{(x, y) \mid y > 2x - 1\}$



51. $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$



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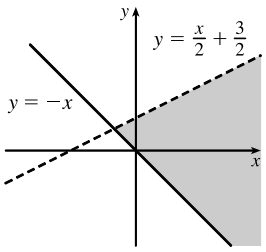
52. $\{(x, y) \mid -x \le y < \frac{1}{2}(x+3)\}$



53. Let $P\,(0, y)$ be a point on the y-axis. The distance from P to $(5, -5)$ is $\sqrt{(5-0)^2 + (-5-y)^2} = \sqrt{5^2 + (y+5)^2}$. The distance from P to $(1, 1)$ is $\sqrt{(1-0)^2 + (1-y)^2} = \sqrt{1^2 + (y-1)^2}$. We want these distances to be equal: $\sqrt{5^2 + (y+5)^2} = \sqrt{1^2 + (y-1)^2} \quad \Leftrightarrow \quad 5^2 + (y+5)^2 = 1^2 + (y-1)^2 \quad \Leftrightarrow$ $25 + (y^2 + 10y + 25) = 1 + (y^2 - 2y + 1) \quad \Leftrightarrow \quad 12y = -48 \quad \Leftrightarrow \quad y = -4$. So the desired point is $(0, -4)$.

54. Let M be the point $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$. Then

$$|MP_1|^2 = \left(x_1 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_1 - \frac{y_1 + y_2}{2}\right)^2 = \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2$$

$$|MP_2|^2 = \left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2 = \left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2$$

Hence, $|MP_1| = |MP_2|$; that is, M is equidistant from P_1 and P_2.

55. (a) Using the midpoint formula from Exercise 54 with $(1, 3)$ and $(7, 15)$, we get $\left(\frac{1+7}{2}, \frac{3+15}{2}\right) = (4, 9)$.

(b) Using the midpoint formula from Exercise 54 with $(-1, 6)$ and $(8, -12)$, we get $\left(\frac{-1+8}{2}, \frac{6+(-12)}{2}\right) = \left(\frac{7}{2}, -3\right)$.

56. With $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$, the midpoint M_1 of AB is $\left(\frac{1+3}{2}, \frac{0+6}{2}\right) = (2, 3)$, the midpoint M_2 of BC is $\left(\frac{3+8}{2}, \frac{6+2}{2}\right) = \left(\frac{11}{2}, 4\right)$, and the midpoint M_3 of CA is $\left(\frac{8+1}{2}, \frac{2+0}{2}\right) = \left(\frac{9}{2}, 1\right)$. The lengths of the medians are

$$|AM_2| = \sqrt{\left(\tfrac{11}{2} - 1\right)^2 + (4-0)^2} = \sqrt{\left(\tfrac{9}{2}\right)^2 + 4^2} = \sqrt{\tfrac{145}{4}} = \tfrac{\sqrt{145}}{2}$$

$$|BM_3| = \sqrt{\left(\tfrac{9}{2} - 3\right)^2 + (1-6)^2} = \sqrt{\left(\tfrac{3}{2}\right)^2 + (-5)^2} = \sqrt{\tfrac{109}{4}} = \tfrac{\sqrt{109}}{2}$$

$$|CM_1| = \sqrt{(2-8)^2 + (3-2)^2} = \sqrt{(-6)^2 + 1^2} = \sqrt{37}$$

57. $2x - y = 4 \quad \Leftrightarrow \quad y = 2x - 4 \quad \Rightarrow \quad m_1 = 2$ and $6x - 2y = 10 \quad \Leftrightarrow \quad 2y = 6x - 10 \quad \Leftrightarrow \quad y = 3x - 5 \quad \Rightarrow \quad m_2 = 3$. Since $m_1 \neq m_2$, the two lines are not parallel. To find the point of intersection: $2x - 4 = 3x - 5 \quad \Leftrightarrow \quad x = 1 \quad \Rightarrow$ $y = -2$. Thus, the point of intersection is $(1, -2)$.

58. $3x - 5y + 19 = 0 \quad \Leftrightarrow \quad 5y = 3x + 19 \quad \Leftrightarrow \quad y = \frac{3}{5}x + \frac{19}{5} \quad \Rightarrow \quad m_1 = \frac{3}{5}$ and $10x + 6y - 50 = 0 \quad \Leftrightarrow$ $6y = -10x + 50 \quad \Leftrightarrow \quad y = -\frac{5}{3}x + \frac{25}{3} \quad \Rightarrow \quad m_2 = -\frac{5}{3}$. Since $m_1 m_2 = \frac{3}{5}\left(-\frac{5}{3}\right) = -1$, the two lines are perpendicular. To find the point of intersection: $\frac{3}{5}x + \frac{19}{5} = -\frac{5}{3}x + \frac{25}{3} \quad \Leftrightarrow \quad 9x + 57 = -25x + 125 \quad \Leftrightarrow \quad 34x = 68 \quad \Leftrightarrow \quad x = 2 \quad \Rightarrow$ $y = \frac{3}{5} \cdot 2 + \frac{19}{5} = \frac{25}{5} = 5$. Thus, the point of intersection is $(2, 5)$.

59. With $A(1, 4)$ and $B(7, -2)$, the slope of segment AB is $\frac{-2-4}{7-1} = -1$, so its perpendicular bisector has slope 1. The midpoint of AB is $\left(\frac{1+7}{2}, \frac{4+(-2)}{2}\right) = (4, 1)$, so an equation of the perpendicular bisector is $y - 1 = 1(x - 4)$ or $y = x - 3$.

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60. (a) Side PQ has slope $\frac{4-0}{3-1}=2$, so its equation is $y-0=2(x-1) \quad \Leftrightarrow \quad y=2x-2$. Side QR has slope $\frac{6-4}{-1-3}=-\frac{1}{2}$, so its equation is $y-4=-\frac{1}{2}(x-3) \quad \Leftrightarrow \quad y=-\frac{1}{2}x+\frac{11}{2}$. Side RP has slope $\frac{0-6}{1-(-1)}=-3$, so its equation is $y-0=-3(x-1) \quad \Leftrightarrow \quad y=-3x+3$.

(b) M_1 (the midpoint of PQ) has coordinates $\left(\frac{1+3}{2}, \frac{0+4}{2}\right)=(2,2)$. M_2 (the midpoint of QR) has coordinates $\left(\frac{3-1}{2}, \frac{4+6}{2}\right)=(1,5)$. M_3 (the midpoint of RP) has coordinates $\left(\frac{1-1}{2}, \frac{0+6}{2}\right)=(0,3)$. RM_1 has slope $\frac{2-6}{2-(-1)}=-\frac{4}{3}$ and hence equation $y-2=-\frac{4}{3}(x-2) \quad \Leftrightarrow \quad y=-\frac{4}{3}x+\frac{14}{3}$. PM_2 is a vertical line with equation $x=1$. QM_3 has slope $\frac{3-4}{0-3}=\frac{1}{3}$ and hence equation $y-3=\frac{1}{3}(x-0) \quad \Leftrightarrow \quad y=\frac{1}{3}x+3$. PM_2 and RM_1 intersect where $x=1$ and $y=-\frac{4}{3}(1)+\frac{14}{3}=\frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$. PM_2 and QM_3 intersect where $x=1$ and $y=\frac{1}{3}(1)+3=\frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$, so this is the point where all three medians intersect.

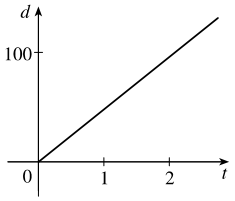
61. (a) Since the x-intercept is a, the point $(a,0)$ is on the line, and similarly since the y-intercept is b, $(0,b)$ is on the line. Hence, the slope of the line is $m=\dfrac{b-0}{0-a}=-\dfrac{b}{a}$. Substituting into $y=mx+b$ gives $y=-\dfrac{b}{a}x+b \quad \Leftrightarrow \quad \dfrac{b}{a}x+y=b \quad \Leftrightarrow \quad \dfrac{x}{a}+\dfrac{y}{b}=1$.

(b) Letting $a=6$ and $b=-8$ gives $\dfrac{x}{6}+\dfrac{y}{-8}=1 \quad \Leftrightarrow \quad -8x+6y=-48$ [multiply by -48] $\quad \Leftrightarrow \quad 6y=8x-48 \quad \Leftrightarrow \quad 3y=4x-24 \quad \Leftrightarrow \quad y=\frac{4}{3}x-8$.

62. (a) Let $d=$ distance traveled (in miles) and $t=$ time elapsed (in hours). At $t=0$, $d=0$ and at $t=50$ minutes $=50\cdot\frac{1}{60}=\frac{5}{6}$ h, $d=40$. Thus, we have two points: $(0,0)$ and $\left(\frac{5}{6}, 40\right)$, so $m=\dfrac{40-0}{5/6-0}=48$ and $d=48t$.

(c) The slope is 48 and represents the car's speed in mi/h.

(b)



C Graphs of Second-Degree Equations

1. An equation of the circle with center $(3,-1)$ and radius 5 is $(x-3)^2+(y+1)^2=5^2=25$.

2. An equation of the circle with center $(-2,-8)$ and radius 10 is $(x+2)^2+(y+8)^2=10^2=100$.

3. The equation has the form $x^2+y^2=r^2$. Since $(4,7)$ lies on the circle, we have $4^2+7^2=r^2 \quad \Rightarrow \quad r^2=65$. So the required equation is $x^2+y^2=65$.

4. The equation has the form $(x+1)^2+(y-5)^2=r^2$. Since $(-4,-6)$ lies on the circle, we have $r^2=(-4+1)^2+(-6-5)^2=130$. So an equation is $(x+1)^2+(y-5)^2=130$.

5. $x^2+y^2-4x+10y+13=0 \quad \Leftrightarrow \quad x^2-4x+y^2+10y=-13 \quad \Leftrightarrow$ $(x^2-4x+4)+(y^2+10y+25)=-13+4+25=16 \quad \Leftrightarrow \quad (x-2)^2+(y+5)^2=4^2$. Thus, we have a circle with center $(2,-5)$ and radius 4.

6. $x^2+y^2+6y+2=0 \quad \Leftrightarrow \quad x^2+(y^2+6y+9)=-2+9 \quad \Leftrightarrow \quad x^2+(y+3)^2=7$. Thus, we have a circle with center $(0,-3)$ and radius $\sqrt{7}$.

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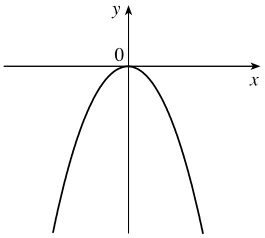
7. $x^2+y^2+x=0 \quad \Leftrightarrow \quad \left(x^2+x+\frac{1}{4}\right)+y^2=\frac{1}{4} \quad \Leftrightarrow \quad \left(x+\frac{1}{2}\right)^2+y^2=\left(\frac{1}{2}\right)^2$. Thus, we have a circle with center $\left(-\frac{1}{2},0\right)$ and radius $\frac{1}{2}$.

8. $16x^2+16y^2+8x+32y+1=0 \quad \Leftrightarrow \quad 16\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+16(y^2+2y+1)=-1+1+16 \quad \Leftrightarrow$ $16\left(x+\frac{1}{4}\right)^2+16(y+1)^2=16 \quad \Leftrightarrow \quad \left(x+\frac{1}{4}\right)^2+(y+1)^2=1$. Thus, we have a circle with center $\left(-\frac{1}{4},-1\right)$ and radius 1.

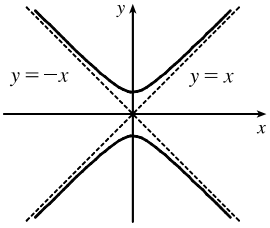
9. $2x^2+2y^2-x+y=1 \quad \Leftrightarrow \quad 2\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)+2\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)=1+\frac{1}{8}+\frac{1}{8} \quad \Leftrightarrow$ $2\left(x-\frac{1}{4}\right)^2+2\left(y+\frac{1}{4}\right)^2=\frac{5}{4} \quad \Leftrightarrow \quad \left(x-\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2=\frac{5}{8}$. Thus, we have a circle with center $\left(\frac{1}{4},-\frac{1}{4}\right)$ and radius $\frac{\sqrt{5}}{2\sqrt{2}}=\frac{\sqrt{10}}{4}$.

10. $x^2+y^2+ax+by+c=0 \quad \Leftrightarrow \quad \left(x^2+ax+\frac{1}{4}a^2\right)+\left(y^2+by+\frac{1}{4}b^2\right)=-c+\frac{1}{4}a^2+\frac{1}{4}b^2 \quad \Leftrightarrow$ $\left(x+\frac{1}{2}a\right)^2+\left(y+\frac{1}{2}b\right)^2=\frac{1}{4}(a^2+b^2-4c)$. For this to represent a nondegenerate circle, $\frac{1}{4}(a^2+b^2-4c)>0$ or $a^2+b^2>4c$. If this condition is satisfied, the circle has center $\left(-\frac{1}{2}a,-\frac{1}{2}b\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2-4c}$.

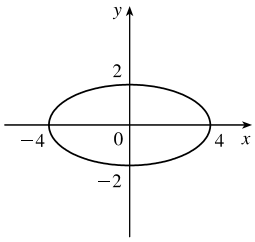
11. $y=-x^2$. Parabola



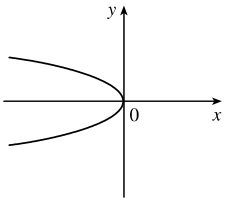
12. $y^2-x^2=1$. Hyperbola



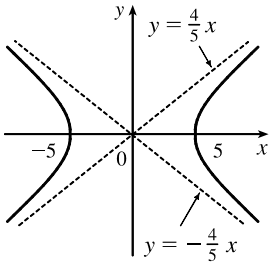
13. $x^2+4y^2=16 \quad \Leftrightarrow \quad \dfrac{x^2}{16}+\dfrac{y^2}{4}=1$. Ellipse



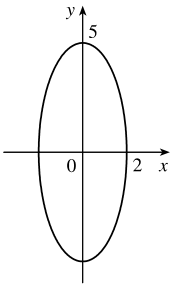
14. $x=-2y^2$. Parabola



15. $16x^2-25y^2=400 \quad \Leftrightarrow \quad \dfrac{x^2}{25}-\dfrac{y^2}{16}=1$. Hyperbola



16. $25x^2+4y^2=100 \quad \Leftrightarrow \quad \dfrac{x^2}{4}+\dfrac{y^2}{25}=1$. Ellipse



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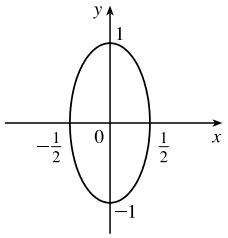
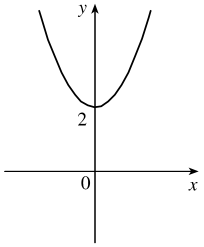
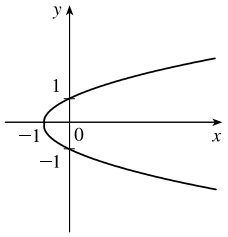
17. $4x^2 + y^2 = 1 \quad \Leftrightarrow \quad \dfrac{x^2}{1/4} + y^2 = 1$. Ellipse



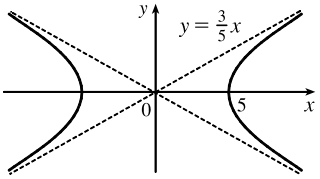
18. $y = x^2 + 2$. Parabola with vertex at $(0, 2)$



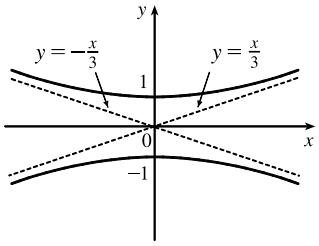
19. $x = y^2 - 1$. Parabola with vertex at $(-1, 0)$



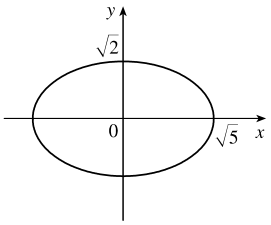
20. $9x^2 - 25y^2 = 225 \quad \Leftrightarrow \quad \dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$. Hyperbola



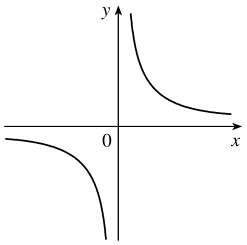
21. $9y^2 - x^2 = 9 \quad \Leftrightarrow \quad y^2 - \dfrac{x^2}{9} = 1$. Hyperbola



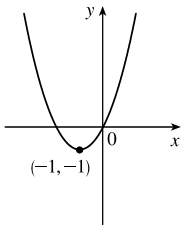
22. $2x^2 + 5y^2 = 10 \quad \Leftrightarrow \quad \dfrac{x^2}{5} + \dfrac{y^2}{2} = 1$. Ellipse



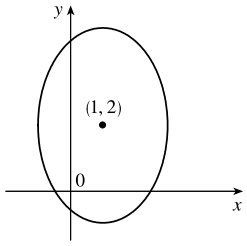
23. $xy = 4$. Hyperbola



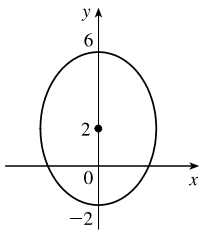
24. $y = x^2 + 2x = \left(x^2 + 2x + 1\right) - 1 = (x + 1)^2 - 1$.
Parabola with vertex at $(-1, -1)$



25. $9(x - 1)^2 + 4(y - 2)^2 = 36 \quad \Leftrightarrow$
$\dfrac{(x - 1)^2}{4} + \dfrac{(y - 2)^2}{9} = 1$. Ellipse centered at $(1, 2)$



26. $16x^2 + 9y^2 - 36y = 108 \quad \Leftrightarrow$
$16x^2 + 9(y^2 - 4y + 4) = 108 + 36 = 144 \quad \Leftrightarrow$
$\dfrac{x^2}{9} + \dfrac{(y - 2)^2}{16} = 1$. Ellipse centered at $(0, 2)$



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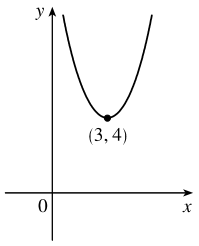
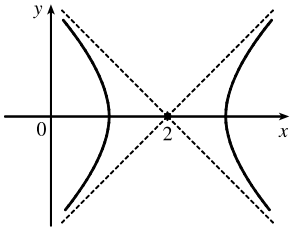
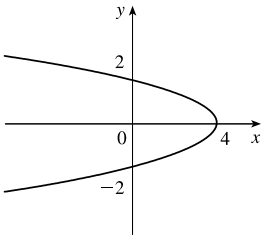
27. $y = x^2 - 6x + 13 = \left(x^2 - 6x + 9\right) + 4 = (x-3)^2 + 4$.
Parabola with vertex at $(3, 4)$



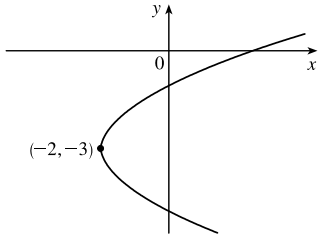
28. $x^2 - y^2 - 4x + 3 = 0 \quad \Leftrightarrow$
$\left(x^2 - 4x + 4\right) - y^2 = -3 + 4 = 1 \quad \Leftrightarrow$
$(x-2)^2 - y^2 = 1$. Hyperbola centered at $(2, 0)$



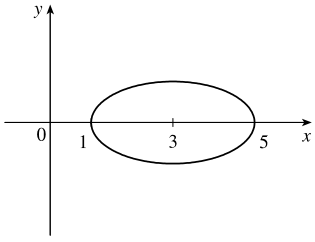
29. $x = 4 - y^2 = -y^2 + 4$. Parabola with vertex at $(4, 0)$



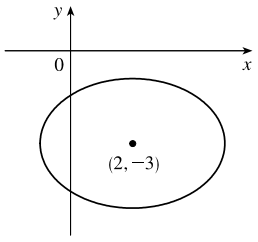
30. $y^2 - 2x + 6y + 5 = 0 \quad \Leftrightarrow \quad y^2 + 6y + 9 = 2x + 4 \quad \Leftrightarrow$
$(y+3)^2 = 2(x+2)$. Parabola with vertex $(-2, -3)$



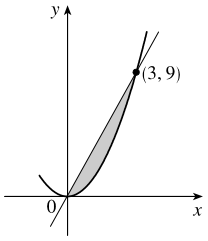
31. $x^2 + 4y^2 - 6x + 5 = 0 \quad \Leftrightarrow$
$(x^2 - 6x + 9) + 4y^2 = -5 + 9 = 4 \quad \Leftrightarrow$
$\dfrac{(x-3)^2}{4} + y^2 = 1$. Ellipse centered at $(3, 0)$



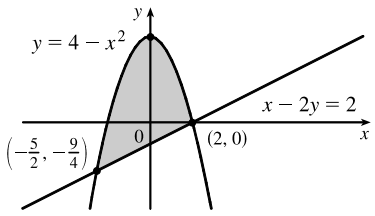
32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0 \quad \Leftrightarrow$
$4(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -61 + 16 + 81 = 36$
$\Leftrightarrow \quad \dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{4} = 1$. Ellipse centered at $(2, -3)$



33. $y = 3x$ and $y = x^2$ intersect where $3x = x^2 \quad \Leftrightarrow$
$0 = x^2 - 3x = x(x-3)$, that is, at $(0, 0)$ and $(3, 9)$.



34. $y = 4 - x^2$, $x - 2y = 2$. Substitute y from the first equation into the second: $x - 2(4 - x^2) = 2 \quad \Leftrightarrow$
$2x^2 + x - 10 = 0 \quad \Leftrightarrow \quad (2x+5)(x-2) = 0 \quad \Leftrightarrow$
$x = -\frac{5}{2}$ or 2. So the points of intersection are $\left(-\frac{5}{2}, -\frac{9}{4}\right)$ and $(2, 0)$.



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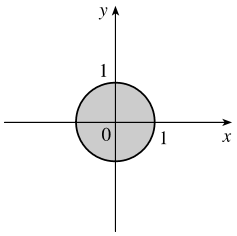
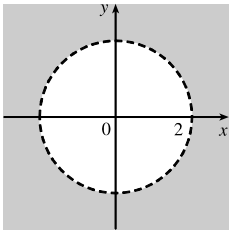
35. The parabola must have an equation of the form $y = a(x-1)^2 - 1$. Substituting $x = 3$ and $y = 3$ into the equation gives $3 = a(3-1)^2 - 1$, so $a = 1$, and the equation is $y = (x-1)^2 - 1 = x^2 - 2x$. Note that using the other point $(-1, 3)$ would have given the same value for a, and hence the same equation.

36. The ellipse has an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Substituting $x = 1$ and $y = -\dfrac{10\sqrt{2}}{3}$ gives $\dfrac{1^2}{a^2} + \dfrac{\left(-10\sqrt{2}/3\right)^2}{b^2} = \dfrac{1}{a^2} + \dfrac{200}{9b^2} = 1$. Substituting $x = -2$ and $y = \dfrac{5\sqrt{5}}{3}$ gives $\dfrac{(-2)^2}{a^2} + \dfrac{\left(5\sqrt{5}/3\right)^2}{b^2} = \dfrac{4}{a^2} + \dfrac{125}{9b^2} = 1$. From the first equation, $\dfrac{1}{a^2} = 1 - \dfrac{200}{9b^2}$. Putting this into the second equation gives $4\left(1 - \dfrac{200}{9b^2}\right) + \dfrac{125}{9b^2} = 1 \quad \Leftrightarrow$ $3 = \dfrac{675}{9b^2} \quad \Leftrightarrow \quad b^2 = \dfrac{675}{27} = 25$, so $b = 5$. Hence $\dfrac{1}{a^2} = 1 - \dfrac{200}{9(5)^2} = \dfrac{1}{9}$ and so $a = 3$. The equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

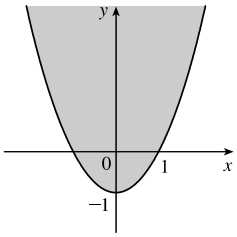
37. $\{(x, y) \mid x^2 + y^2 \le 1\}$



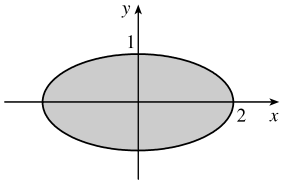
38. $\{(x, y) \mid x^2 + y^2 > 4\}$



39. $\{(x, y) \mid y \ge x^2 - 1\}$



40. $\{(x, y) \mid x^2 + 4y^2 \le 4\}$



D Trigonometry

1. $210^\circ = 210^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{7\pi}{6}$ rad

2. $300^\circ = 300^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{3}$ rad

3. $9^\circ = 9^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{20}$ rad

4. $-315^\circ = -315^\circ\left(\frac{\pi}{180^\circ}\right) = -\frac{7\pi}{4}$ rad

5. $900^\circ = 900^\circ\left(\frac{\pi}{180^\circ}\right) = 5\pi$ rad

6. $36^\circ = 36^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{5}$ rad

7. $4\pi \text{ rad} = 4\pi\left(\frac{180^\circ}{\pi}\right) = 720^\circ$

8. $-\frac{7\pi}{2} \text{ rad} = -\frac{7\pi}{2}\left(\frac{180^\circ}{\pi}\right) = -630^\circ$

9. $\frac{5\pi}{12} \text{ rad} = \frac{5\pi}{12}\left(\frac{180^\circ}{\pi}\right) = 75^\circ$

10. $\frac{8\pi}{3} \text{ rad} = \frac{8\pi}{3}\left(\frac{180^\circ}{\pi}\right) = 480^\circ$

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11. $-\frac{3\pi}{8}$ rad $= -\frac{3\pi}{8}\left(\frac{180^\circ}{\pi}\right) = -67.5^\circ$

12. 5 rad $= 5\left(\frac{180^\circ}{\pi}\right) = \left(\frac{900}{\pi}\right)^\circ$

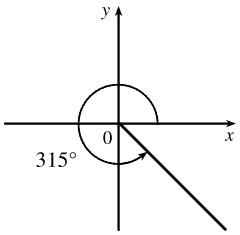
13. Using Formula 3, $a = r\theta = 36 \cdot \frac{\pi}{12} = 3\pi$ cm.

14. Using Formula 3, $a = r\theta = 10 \cdot 72^\circ\left(\frac{\pi}{180^\circ}\right) = 4\pi$ cm.

15. Using Formula 3, $\theta = a/r = \frac{1}{1.5} = \frac{2}{3}$ rad $= \frac{2}{3}\left(\frac{180^\circ}{\pi}\right) = \left(\frac{120}{\pi}\right)^\circ \approx 38.2^\circ$.

16. $a = r\theta \quad \Rightarrow \quad r = \frac{a}{\theta} = \frac{6}{3\pi/4} = \frac{8}{\pi}$ cm

17.



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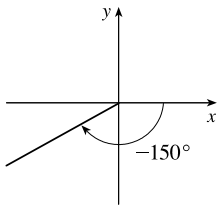

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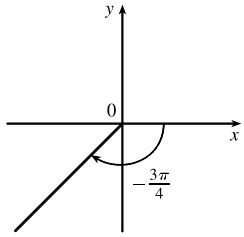
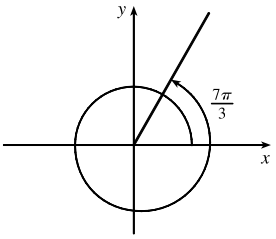


20.



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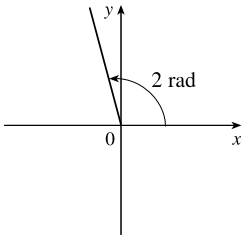

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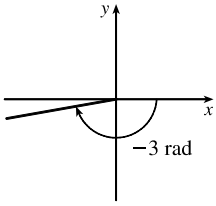

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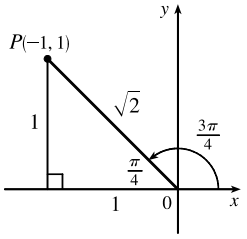
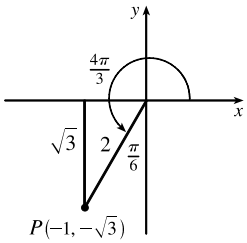


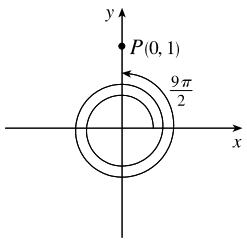
From the diagram we see that a point on the terminal side is $P(-1, 1)$. Therefore, taking $x = -1$, $y = 1$, $r = \sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan\frac{3\pi}{4} = -1$, $\csc\frac{3\pi}{4} = \sqrt{2}$, $\sec\frac{3\pi}{4} = -\sqrt{2}$, and $\cot\frac{3\pi}{4} = -1$.

24.



From the diagram and Figure 9, we see that a point on the terminal side is $P\left(-1, -\sqrt{3}\right)$. Therefore, taking $x = -1$, $y = -\sqrt{3}$, $r = 2$ in the definitions of the trigonometric ratios, we have $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = -\frac{1}{2}$, $\tan\frac{4\pi}{3} = \sqrt{3}$, $\csc\frac{4\pi}{3} = -\frac{2}{\sqrt{3}}$, $\sec\frac{4\pi}{3} = -2$, and $\cot\frac{4\pi}{3} = \frac{1}{\sqrt{3}}$.

25.



From the diagram we see that a point on the terminal side is $P(0, 1)$. Therefore taking $x = 0$, $y = 1$, $r = 1$ in the definitions of the trigonometric ratios, we have $\sin\frac{9\pi}{2} = 1$, $\cos\frac{9\pi}{2} = 0$, $\tan\frac{9\pi}{2} = y/x$ is undefined since $x = 0$, $\csc\frac{9\pi}{2} = 1$, $\sec\frac{9\pi}{2} = r/x$ is undefined since $x = 0$, and $\cot\frac{9\pi}{2} = 0$.

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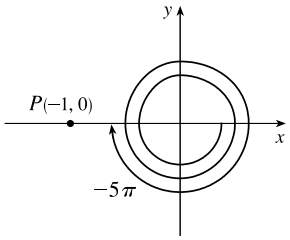
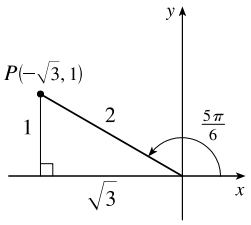
26.



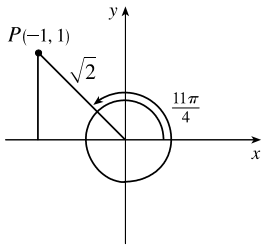
From the diagram, we see that a point on the terminal side is $P(-1,0)$. Therefore taking $x=-1$, $y=0$, $r=1$ in the definitions of the trigonometric ratios we have $\sin(-5\pi)=0$, $\cos(-5\pi)=-1$, $\tan(-5\pi)=0$, $\csc(-5\pi)$ is undefined, $\sec(-5\pi)=-1$, and $\cot(-5\pi)$ is undefined.

27.



Using Figure 9 we see that a point on the terminal side is $P\left(-\sqrt{3},1\right)$. Therefore taking $x=-\sqrt{3}$, $y=1$, $r=2$ in the definitions of the trigonometric ratios, we have $\sin\frac{5\pi}{6}=\frac{1}{2}$, $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$, $\tan\frac{5\pi}{6}=-\frac{1}{\sqrt{3}}$, $\csc\frac{5\pi}{6}=2$, $\sec\frac{5\pi}{6}=-\frac{2}{\sqrt{3}}$, and $\cot\frac{5\pi}{6}=-\sqrt{3}$.

28.



From the diagram, we see that a point on the terminal side is $P(-1,1)$. Therefore taking $x=-1$, $y=1$, $r=\sqrt{2}$ in the definitions of the trigonometric ratios we have $\sin\frac{11\pi}{4}=\frac{1}{\sqrt{2}}$, $\cos\frac{11\pi}{4}=-\frac{1}{\sqrt{2}}$, $\tan\frac{11\pi}{4}=-1$, $\csc\frac{11\pi}{4}=\sqrt{2}$, $\sec\frac{11\pi}{4}=-\sqrt{2}$, and $\cot\frac{11\pi}{4}=-1$.

29. $\sin\theta=y/r=\frac{3}{5}\quad\Rightarrow\quad y=3$, $r=5$, and $x=\sqrt{r^2-y^2}=4$ (since $0<\theta<\frac{\pi}{2}$). Therefore taking $x=4$, $y=3$, $r=5$ in the definitions of the trigonometric ratios, we have $\cos\theta=\frac{4}{5}$, $\tan\theta=\frac{3}{4}$, $\csc\theta=\frac{5}{3}$, $\sec\theta=\frac{5}{4}$, and $\cot\theta=\frac{4}{3}$.

30. Since $0<\alpha<\frac{\pi}{2}$, α is in the first quadrant where x and y are both positive. Therefore, $\tan\alpha=y/x=\frac{2}{1}\quad\Rightarrow\quad y=2$, $x=1$, and $r=\sqrt{x^2+y^2}=\sqrt{5}$. Taking $x=1$, $y=2$, $r=\sqrt{5}$ in the definitions of the trigonometric ratios, we have $\sin\alpha=\frac{2}{\sqrt{5}}$, $\cos\alpha=\frac{1}{\sqrt{5}}$, $\csc\alpha=\frac{\sqrt{5}}{2}$, $\sec\alpha=\sqrt{5}$, and $\cot\alpha=\frac{1}{2}$.

31. $\frac{\pi}{2}<\phi<\pi\quad\Rightarrow\quad\phi$ is in the second quadrant, where x is negative and y is positive. Therefore $\sec\phi=r/x=-1.5=-\frac{3}{2}\quad\Rightarrow\quad r=3$, $x=-2$, and $y=\sqrt{r^2-x^2}=\sqrt{5}$. Taking $x=-2$, $y=\sqrt{5}$, and $r=3$ in the definitions of the trigonometric ratios, we have $\sin\phi=\frac{\sqrt{5}}{3}$, $\cos\phi=-\frac{2}{3}$, $\tan\phi=-\frac{\sqrt{5}}{2}$, $\csc\phi=\frac{3}{\sqrt{5}}$, and $\cot\theta=-\frac{2}{\sqrt{5}}$.

32. Since $\pi<x<\frac{3\pi}{2}$, x is in the third quadrant where x and y are both negative. Therefore $\cos x=x/r=-\frac{1}{3}\quad\Rightarrow\quad x=-1$, $r=3$, and $y=-\sqrt{r^2-x^2}=-\sqrt{8}=-2\sqrt{2}$. Taking $x=-1$, $r=3$, $y=-2\sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin x=-\frac{2\sqrt{2}}{3}$, $\tan x=2\sqrt{2}$, $\csc x=-\frac{3}{2\sqrt{2}}$, $\sec x=-3$, and $\cot x=\frac{1}{2\sqrt{2}}$.

33. $\pi<\beta<2\pi$ means that β is in the third or fourth quadrant where y is negative. Also since $\cot\beta=x/y=3$ which is positive, x must also be negative. Therefore $\cot\beta=x/y=\frac{3}{1}\quad\Rightarrow\quad x=-3$, $y=-1$, and $r=\sqrt{x^2+y^2}=\sqrt{10}$. Taking $x=-3$, $y=-1$ and $r=\sqrt{10}$ in the definitions of the trigonometric ratios, we have $\sin\beta=-\frac{1}{\sqrt{10}}$, $\cos\beta=-\frac{3}{\sqrt{10}}$, $\tan\beta=\frac{1}{3}$, $\csc\beta=-\sqrt{10}$, and $\sec\beta=-\frac{\sqrt{10}}{3}$.

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34. Since $\frac{3\pi}{2} < \theta < 2\pi$, θ is in the fourth quadrant where x is positive and y is negative. Therefore $\csc\theta = r/y = -\frac{4}{3} \Rightarrow r = 4$, $y = -3$, and $x = \sqrt{r^2 - y^2} = \sqrt{7}$. Taking $x = \sqrt{7}$, $y = -3$, and $r = 4$ in the definitions of the trigonometric ratios, we have $\sin\theta = -\frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$, $\sec\theta = \frac{4}{\sqrt{7}}$, and $\cot\theta = -\frac{\sqrt{7}}{3}$.

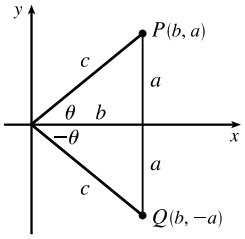
35. $\sin 35^\circ = \dfrac{x}{10} \quad \Rightarrow \quad x = 10\sin 35^\circ \approx 5.73576$ cm

36. $\cos 40^\circ = \dfrac{x}{25} \quad \Rightarrow \quad x = 25\cos 40^\circ \approx 19.15111$ cm

37. $\tan\frac{2\pi}{5} = \dfrac{x}{8} \quad \Rightarrow \quad x = 8\tan\frac{2\pi}{5} \approx 24.62147$ cm

38. $\cos\dfrac{3\pi}{8} = \dfrac{22}{x} \quad \Rightarrow \quad x = \dfrac{22}{\cos\frac{3\pi}{8}} \approx 57.48877$ cm

39.



(a) From the diagram we see that $\sin\theta = \dfrac{y}{r} = \dfrac{a}{c}$, and $\sin(-\theta) = \dfrac{-a}{c} = -\dfrac{a}{c} = -\sin\theta$.

(b) Again from the diagram we see that $\cos\theta = \dfrac{x}{r} = \dfrac{b}{c} = \cos(-\theta)$.

40. (a) Using (13a) and (13b), we have

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\,\cos y + \cos x\,\sin y}{\cos x\,\cos y - \sin x\,\sin y} = \frac{\dfrac{\sin x\,\cos y}{\cos x\,\cos y} + \dfrac{\cos x\,\sin y}{\cos x\,\cos y}}{\dfrac{\cos x\,\cos y}{\cos x\,\cos y} - \dfrac{\sin x\,\sin y}{\cos x\,\cos y}} = \frac{\tan x + \tan y}{1 - \tan x\,\tan y}$$

(b) From (11a) and (11b), we have $\tan(-\theta) = -\tan\theta$, so (15a) implies that

$$\tan(x-y) = \tan(x+(-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x\,\tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x\,\tan y}$$

41. (a) Using (13a) and (14a), we have

$$\tfrac{1}{2}[\sin(x+y) + \sin(x-y)] = \tfrac{1}{2}[\sin x\,\cos y + \cos x\,\sin y + \sin x\,\cos y - \cos x\,\sin y] = \tfrac{1}{2}(2\sin x\,\cos y) = \sin x\,\cos y.$$

(b) This time, using (13b) and (14b), we have

$$\tfrac{1}{2}[\cos(x+y) + \cos(x-y)] = \tfrac{1}{2}[\cos x\,\cos y - \sin x\,\sin y + \cos x\,\cos y + \sin x\,\sin y] = \tfrac{1}{2}(2\cos x\,\cos y) = \cos x\,\cos y.$$

(c) Again using (13b) and (14b), we have

$$\begin{aligned}\tfrac{1}{2}[\cos(x-y) - \cos(x+y)] &= \tfrac{1}{2}[\cos x\,\cos y + \sin x\,\sin y - \cos x\,\cos y + \sin x\,\sin y] \\ &= \tfrac{1}{2}(2\sin x\,\sin y) = \sin x\,\sin y\end{aligned}$$

42. Using (14b), $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = 0\cdot\cos x + 1\cdot\sin x = \sin x$.

43. Using (13a), we have $\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x = 1\cdot\cos x + 0\cdot\sin x = \cos x$.

44. Using (14a), we have $\sin(\pi - x) = \sin\pi\cos x - \cos\pi\,\sin x = 0\cdot\cos x - (-1)\sin x = \sin x$.

45. Using (7), we have $\sin\theta\cot\theta = \sin\theta\cdot\dfrac{\cos\theta}{\sin\theta} = \cos\theta$.

46. $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\,\cos x + \cos^2 x = (\sin^2 x + \cos^2 x) + \sin 2x$ [by (16a)] $= 1 + \sin 2x$ [by (8)]

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47. $\sec y - \cos y = \dfrac{1}{\cos y} - \cos y$ [by (7)] $= \dfrac{1-\cos^2 y}{\cos y} = \dfrac{\sin^2 y}{\cos y}$ [by (8)] $= \dfrac{\sin y}{\cos y}\sin y = \tan y \sin y$ [by (7)]

48. $\tan^2\alpha - \sin^2\alpha = \dfrac{\sin^2\alpha}{\cos^2\alpha} - \sin^2\alpha = \dfrac{\sin^2\alpha - \sin^2\alpha\cos^2\alpha}{\cos^2\alpha} = \dfrac{\sin^2\alpha(1-\cos^2\alpha)}{\cos^2\alpha} = \tan^2\alpha\sin^2\alpha$ [by (7), (8)]

49. $\cot^2\theta + \sec^2\theta = \dfrac{\cos^2\theta}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}$ [by (7)] $= \dfrac{\cos^2\theta\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$

$= \dfrac{(1-\sin^2\theta)(1-\sin^2\theta)+\sin^2\theta}{\sin^2\theta\cos^2\theta}$ [by (8)] $= \dfrac{1-\sin^2\theta+\sin^4\theta}{\sin^2\theta\cos^2\theta}$

$= \dfrac{\cos^2\theta+\sin^4\theta}{\sin^2\theta\cos^2\theta}$ [by (7)] $= \dfrac{1}{\sin^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \csc^2\theta + \tan^2\theta$ [by (7)]

50. $2\csc 2t = \dfrac{2}{\sin 2t} = \dfrac{2}{2\sin t\cos t}$ [by (16a)] $= \dfrac{1}{\sin t\cos t} = \sec t\csc t$

51. Using (15a), we have $\tan 2\theta = \tan(\theta+\theta) = \dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

52. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = \dfrac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \dfrac{2}{1-\sin^2\theta} = \dfrac{2}{\cos^2\theta}$ [by (8)] $= 2\sec^2\theta$

53. Using (16a) and (17a),

$$\begin{aligned}\sin x\sin 2x + \cos x\cos 2x &= \sin x(2\sin x\cos x) + \cos x(2\cos^2 x - 1) = 2\sin^2 x\cos x + 2\cos^3 x - \cos x\\ &= 2(1-\cos^2 x)\cos x + 2\cos^3 x - \cos x \quad \text{[by (8)]}\\ &= 2\cos x - 2\cos^3 x + 2\cos^3 x - \cos x = \cos x\end{aligned}$$

Or: $\sin x\sin 2x + \cos x\cos 2x = \cos(2x-x)$ [by 14(b)] $= \cos x$

54. We start with the right side using equations (13a) and (14a):

$$\begin{aligned}\sin(x+y)\sin(x-y) &= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)\\ &= \sin^2 x\cos^2 y - \sin x\cos y\cos x\sin y + \cos x\sin y\sin x\cos y - \cos^2 x\sin^2 y\\ &= \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y \quad \text{[by (8)]}\\ &= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y = \sin^2 x - \sin^2 y\end{aligned}$$

55. $\dfrac{\sin\phi}{1-\cos\phi} = \dfrac{\sin\phi}{1-\cos\phi}\cdot\dfrac{1+\cos\phi}{1+\cos\phi} = \dfrac{\sin\phi(1+\cos\phi)}{1-\cos^2\phi} = \dfrac{\sin\phi(1+\cos\phi)}{\sin^2\phi}$ [by (8)]

$= \dfrac{1+\cos\phi}{\sin\phi} = \dfrac{1}{\sin\phi} + \dfrac{\cos\phi}{\sin\phi} = \csc\phi + \cot\phi$ [by (7)]

56. $\tan x + \tan y = \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} = \dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y} = \dfrac{\sin(x+y)}{\cos x\cos y}$ [by (13a)]

57. Using (13a),

$$\begin{aligned}\sin 3\theta + \sin\theta &= \sin(2\theta+\theta) + \sin\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta + \sin\theta\\ &= \sin 2\theta\cos\theta + (2\cos^2\theta - 1)\sin\theta + \sin\theta \quad \text{[by (17a)]}\\ &= \sin 2\theta\cos\theta + 2\cos^2\theta\sin\theta - \sin\theta + \sin\theta = \sin 2\theta\cos\theta + \sin 2\theta\cos\theta \quad \text{[by (16a)]}\\ &= 2\sin 2\theta\cos\theta\end{aligned}$$

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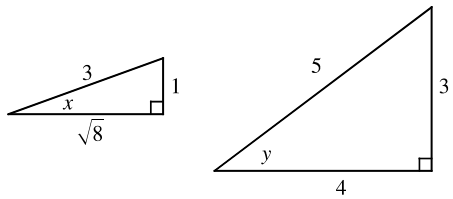
58. We use (13b) with $x = 2\theta$, $y = \theta$ to get

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta\,\cos\theta - \sin 2\theta\,\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\,\cos\theta \quad \text{[by (17a) and (16a)]} \\ &= (2\cos^2\theta - 1)\cos\theta - 2(1 - \cos^2\theta)\,\cos\theta \quad \text{[by (8)]} \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta\end{aligned}$$

59. Since $\sin x = \frac{1}{3}$ we can label the opposite side as having length 1, the hypotenuse as having length 3, and use the Pythagorean Theorem to get that the adjacent side has length $\sqrt{8}$. Then, from the diagram, $\cos x = \frac{\sqrt{8}}{3}$. Similarly we have that $\sin y = \frac{3}{5}$. Now use (13a):



$\sin(x + y) = \sin x\,\cos y + \cos x\,\sin y = \frac{1}{3}\cdot\frac{4}{5} + \frac{\sqrt{8}}{3}\cdot\frac{3}{5} = \frac{4}{15} + \frac{3\sqrt{8}}{15} = \frac{4+6\sqrt{2}}{15}$.

60. Use (13b) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59 to get

$$\cos(x + y) = \cos x\,\cos y - \sin x\,\sin y = \frac{\sqrt{8}}{3}\cdot\frac{4}{5} - \frac{1}{3}\cdot\frac{3}{5} = \frac{8\sqrt{2}-3}{15}$$

61. Using (14b) and the values for $\cos x$ and $\sin y$ obtained in Exercise 59, we have

$$\cos(x - y) = \cos x\,\cos y + \sin x\,\sin y = \frac{\sqrt{8}}{3}\cdot\frac{4}{5} + \frac{1}{3}\cdot\frac{3}{5} = \frac{8\sqrt{2}+3}{15}$$

62. Using (14a) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59, we get

$$\sin(x - y) = \sin x\,\cos y - \cos x\,\sin y = \frac{1}{3}\cdot\frac{4}{5} - \frac{\sqrt{8}}{3}\cdot\frac{3}{5} = \frac{4-6\sqrt{2}}{15}$$

63. Using (16a) and the values for $\sin y$ and $\cos y$ obtained in Exercise 59, we have $\sin 2y = 2\sin y\,\cos y = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$.

64. Using (17a) with $\cos y = \frac{4}{5}$, we have $\cos 2y = 2\cos^2 y - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$.

65. $2\cos x - 1 = 0 \;\Leftrightarrow\; \cos x = \frac{1}{2} \;\Rightarrow\; x = \frac{\pi}{3}, \frac{5\pi}{3}$ for $x \in [0, 2\pi]$.

66. $3\cot^2 x = 1 \;\Leftrightarrow\; 3 = 1/\cot^2 x \;\Leftrightarrow\; \tan^2 x = 3 \;\Leftrightarrow\; \tan x = \pm\sqrt{3} \;\Rightarrow\; x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

67. $2\sin^2 x = 1 \;\Leftrightarrow\; \sin^2 x = \frac{1}{2} \;\Leftrightarrow\; \sin x = \pm\frac{1}{\sqrt{2}} \;\Rightarrow\; x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

68. $|\tan x| = 1 \;\Leftrightarrow\; \tan x = -1$ or $\tan x = 1 \;\Leftrightarrow\; x = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

69. Using (16a), we have $\sin 2x = \cos x \;\Leftrightarrow\; 2\sin x\,\cos x - \cos x = 0 \;\Leftrightarrow\; \cos x(2\sin x - 1) = 0 \;\Leftrightarrow\; \cos x = 0$ or $2\sin x - 1 = 0 \;\Rightarrow\; x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = \frac{1}{2} \;\Rightarrow\; x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Therefore, the solutions are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

70. By (16a), $2\cos x + \sin 2x = 0 \;\Leftrightarrow\; 2\cos x + 2\sin x\cos x = 0 \;\Leftrightarrow\; 2\cos x\,(1 + \sin x) = 0 \;\Leftrightarrow\; \cos x = 0$ or $1 + \sin x = 0 \;\Leftrightarrow\; x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = -1 \;\Rightarrow\; x = \frac{3}{2}\pi$. So the solutions are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

71. $\sin x = \tan x \;\Leftrightarrow\; \sin x - \tan x = 0 \;\Leftrightarrow\; \sin x - \dfrac{\sin x}{\cos x} = 0 \;\Leftrightarrow\; \sin x\left(1 - \dfrac{1}{\cos x}\right) = 0 \;\Leftrightarrow\; \sin x = 0$ or $1 - \dfrac{1}{\cos x} = 0 \;\Rightarrow\; x = 0, \pi, 2\pi$ or $1 = \dfrac{1}{\cos x} \;\Rightarrow\; \cos x = 1 \;\Rightarrow\; x = 0, 2\pi$. Therefore the solutions are $x = 0, \pi, 2\pi$.

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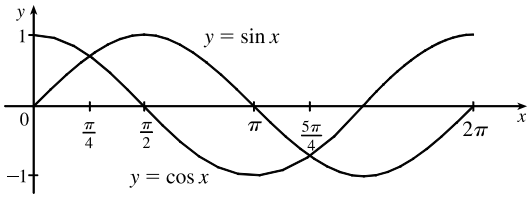
72. By (17a), $2+\cos 2x = 3\cos x \quad \Leftrightarrow \quad 2+2\cos^2 x - 1 = 3\cos x \quad \Leftrightarrow \quad 2\cos^2 x - 3\cos x + 1 = 0 \quad \Leftrightarrow$ $(2\cos x - 1)(\cos x - 1) = 0 \quad \Leftrightarrow \quad \cos x = 1$ or $\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 0, 2\pi$ or $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

73. We know that $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, and from Figure 18(a), we see that $\sin x \le \frac{1}{2} \quad \Rightarrow \quad 0 \le x \le \frac{\pi}{6}$ or $\frac{5\pi}{6} \le x \le 2\pi$ for $x \in [0, 2\pi]$.

74. $2\cos x + 1 > 0 \quad \Rightarrow \quad 2\cos x > -1 \quad \Rightarrow \quad \cos x > -\frac{1}{2}$. $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ and from Figure 18(b), we see that $\cos\, x > -\frac{1}{2}$ when $0 \le x < \frac{2\pi}{3}, \frac{4\pi}{3} < x \le 2\pi$.

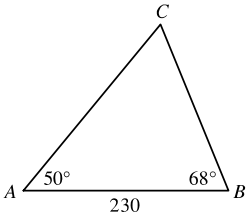
75. $\tan x = -1$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, and $\tan x = 1$ when $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. From Figure 19(a) we see that $-1 < \tan x < 1 \quad \Rightarrow$ $0 \le x < \frac{\pi}{4}, \frac{3\pi}{4} < x < \frac{5\pi}{4}$, and $\frac{7\pi}{4} < x \le 2\pi$.

76. We know that $\sin x = \cos x$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$, and from the diagram we see that $\sin x > \cos x$ when $\frac{\pi}{4} < x < \frac{5\pi}{4}$.



77. $\angle C = 180^\circ - 50^\circ - 68^\circ = 62^\circ$.

Then $a = \dfrac{230\sin 50^\circ}{\sin 62^\circ} \approx 200$ and $b = \dfrac{230\sin 68^\circ}{\sin 62^\circ} \approx 242$.



78. $c^2 = 3^2 + 4^2 - 2\cdot 3\cdot 4\cdot \cos 53^\circ = 9 + 16 - 24\cos 53^\circ \approx 10.556 \quad \Leftrightarrow \quad c \approx \sqrt{10.556} \approx 3.2$. Then

$\sin B = \dfrac{4\sin 53^\circ}{3.25} \approx 0.983 \quad \Leftrightarrow \quad \angle B \approx \sin^{-1}(0.983) \approx 79^\circ$ and $\angle A \approx 180^\circ - 53^\circ - 79^\circ = 48^\circ$.

79. $|AB|^2 = |AC|^2 + |BC|^2 - 2\,|AC|\,|BC|\cos\angle C = (820)^2 + (910)^2 - 2(820)(910)\cos 103^\circ \approx 1{,}836{,}217 \quad \Rightarrow$ $|AB| \approx 1355$ m

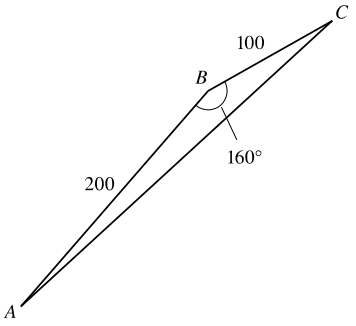
80. By the Law of Cosines we have

$b^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cos 160^\circ \approx 87{,}587.70$. Thus $b \approx 295.95$.

To find $\angle A$ we first use the Law of Sines:

$\dfrac{\sin A}{100} \approx \dfrac{\sin 160^\circ}{295.95} \quad \Leftrightarrow \quad \sin A = 100\cdot\dfrac{\sin 160^\circ}{295.95} \approx 0.11557 \quad \Leftrightarrow$

$A = \sin^{-1}(0.11557) \approx 6.636^\circ$



81. Using the formula from (6), the area of the triangle is $\frac{1}{2}(10)(3)\sin 107^\circ \approx 14.34457\ \text{cm}^2$.

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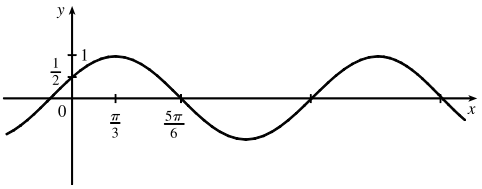
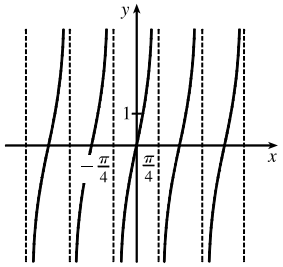
82. By Heron's formula we have $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(4+5+7) = 8$. So then, $\mathcal{A} = \sqrt{8(8-4)(8-5)(8-7)} = 4\sqrt{6} \approx 9.80$.

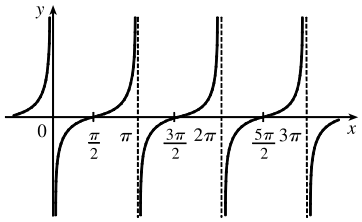
83. $y = \cos\left(x - \frac{\pi}{3}\right)$. We start with the graph of $y = \cos x$ and shift it $\frac{\pi}{3}$ units to the right.



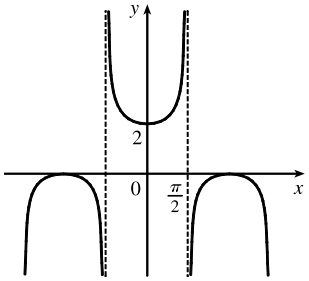
84. $y = \tan 2x$. Start with the graph of $y = \tan x$ with period π and compress it to a period of $\frac{\pi}{2}$.



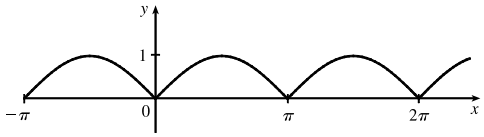
85. $y = \frac{1}{3}\tan\left(x - \frac{\pi}{2}\right)$. We start with the graph of $y = \tan x$, shift it $\frac{\pi}{2}$ units to the right and compress it to $\frac{1}{3}$ of its original vertical size.



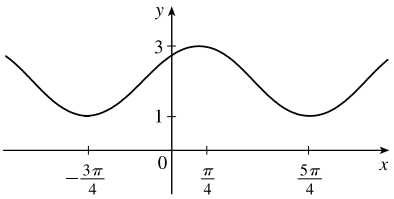
86. $y = 1 + \sec x$. Start with the graph of $y = \sec x$ and raise it by one unit.



87. $y = |\sin x|$. We start with the graph of $y = \sin x$ and reflect the parts below the x-axis about the x-axis.



88. $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$. Start with the graph of $y = \sin x$, and shift it $\frac{\pi}{4}$ units to the left and 2 units up.



89. Using the Law of Cosines, we have $c^2 = 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta) = 2[1 - \cos(\alpha - \beta)]$. Now, using the distance formula, $c^2 = |AB|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$. Equating these two expressions for c^2, we get

$2[1 - \cos(\alpha - \beta)] = \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \quad \Rightarrow$

$1 - \cos(\alpha - \beta) = 1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \Rightarrow \quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

90. $\cos(x + y) = \cos(x - (-y)) = \cos x\cos(-y) + \sin x\sin(-y)$

$= \cos x\cos y - \sin x\sin y$ [using Equations (11a) and (11b)]

91. In Exercise 90 we used the subtraction formula for cosine to prove the addition formula for cosine. Using that formula with $x = \frac{\pi}{2} - \alpha$, $y = \beta$, we get $\cos\left[\left(\frac{\pi}{2} - \alpha\right) + \beta\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \quad \Rightarrow$

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$\cos\left[\frac{\pi}{2}-(\alpha-\beta)\right]=\cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$. Now we use the identities given in the problem, $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$, to get $\sin(\alpha-\beta)=\sin\alpha\,\cos\beta-\cos\alpha\,\sin\beta$.

E Sigma Notation

1. $\sum_{i=1}^{5}\sqrt{i}=\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}$

2. $\sum_{i=1}^{6}\frac{1}{i+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$

3. $\sum_{i=4}^{6}3^i=3^4+3^5+3^6$

4. $\sum_{i=4}^{6}i^3=4^3+5^3+6^3$

5. $\sum_{k=0}^{4}\frac{2k-1}{2k+1}=-1+\frac{1}{3}+\frac{3}{5}+\frac{5}{7}+\frac{7}{9}$

6. $\sum_{k=5}^{8}x^k=x^5+x^6+x^7+x^8$

7. $\sum_{i=1}^{n}i^{10}=1^{10}+2^{10}+3^{10}+\cdots+n^{10}$

8. $\sum_{j=n}^{n+3}j^2=n^2+(n+1)^2+(n+2)^2+(n+3)^2$

9. $\sum_{j=0}^{n-1}(-1)^j=1-1+1-1+\cdots+(-1)^{n-1}$

10. $\sum_{i=1}^{n}f(x_i)\,\Delta x_i=f(x_1)\,\Delta x_1+f(x_2)\,\Delta x_2+f(x_3)\,\Delta x_3+\cdots+f(x_n)\,\Delta x_n$

11. $1+2+3+4+\cdots+10=\sum_{i=1}^{10}i$

12. $\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}+\sqrt{7}=\sum_{i=3}^{7}\sqrt{i}$

13. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{19}{20}=\sum_{i=1}^{19}\frac{i}{i+1}$

14. $\frac{3}{7}+\frac{4}{8}+\frac{5}{9}+\frac{6}{10}+\cdots+\frac{23}{27}=\sum_{i=3}^{23}\frac{i}{i+4}$

15. $2+4+6+8+\cdots+2n=\sum_{i=1}^{n}2i$

16. $1+3+5+7+\cdots+(2n-1)=\sum_{i=1}^{n}(2i-1)$

17. $1+2+4+8+16+32=\sum_{i=0}^{5}2^i$

18. $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}=\sum_{i=1}^{6}\frac{1}{i^2}$

19. $x+x^2+x^3+\cdots+x^n=\sum_{i=1}^{n}x^i$

20. $1-x+x^2-x^3+\cdots+(-1)^n\,x^n=\sum_{i=0}^{n}(-1)^i x^i$

21. $\sum_{i=4}^{8}(3i-2)=[3(4)-2]+[3(5)-2]+[3(6)-2]+[3(7)-2]+[3(8)-2]=10+13+16+19+22=80$

22. $\sum_{i=3}^{6}i(i+2)=3\cdot5+4\cdot6+5\cdot7+6\cdot8=15+24+35+48=122$

23. $\sum_{j=1}^{6}3^{j+1}=3^2+3^3+3^4+3^5+3^6+3^7=9+27+81+243+729+2187=3276$

(For a more general method, see Exercise 47.)

24. $\sum_{k=0}^{8}\cos k\pi=\cos0+\cos\pi+\cos2\pi+\cos3\pi+\cos4\pi+\cos5\pi+\cos6\pi+\cos7\pi+\cos8\pi$

$=1-1+1-1+1-1+1-1+1=1$

25. $\sum_{n=1}^{20}(-1)^n=-1+1-1+1-1+1-1+1-1+1-1+1-1+1-1+1-1+1-1+1-1+1=0$

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26. $\sum_{i=1}^{100} 4 = \underbrace{4+4+4+\cdots+4}_{\text{(100 summands)}} = 100 \cdot 4 = 400$

27. $\sum_{i=0}^{4} (2^i + i^2) = (1+0) + (2+1) + (4+4) + (8+9) + (16+16) = 61$

28. $\sum_{i=-2}^{4} 2^{3-i} = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} = 63.5$

29. $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = 2 \cdot \dfrac{n(n+1)}{2}$ [by Theorem 3(c)] $= n(n+1)$

30. $\sum_{i=1}^{n} (2-5i) = \sum_{i=1}^{n} 2 - \sum_{i=1}^{n} 5i = 2n - 5\sum_{i=1}^{n} i = 2n - \dfrac{5n(n+1)}{2} = \dfrac{4n}{2} - \dfrac{5n^2+5n}{2} = -\dfrac{n(5n+1)}{2}$

31. $\sum_{i=1}^{n} (i^2+3i+4) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4 = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{3n(n+1)}{2} + 4n$

$= \frac{1}{6}[(2n^3+3n^2+n) + (9n^2+9n) + 24n] = \frac{1}{6}(2n^3+12n^2+34n) = \frac{1}{3}n(n^2+6n+17)$

32. $\sum_{i=1}^{n} (3+2i)^2 = \sum_{i=1}^{n} (9+12i+4i^2) = \sum_{i=1}^{n} 9 + 12\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2 = 9n + 6n(n+1) + \dfrac{2n(n+1)(2n+1)}{3}$

$= \dfrac{27n+18n^2+18n+4n^3+6n^2+2n}{3} = \frac{1}{3}(4n^3+24n^2+47n) = \frac{1}{3}n(4n^2+24n+47)$

33. $\sum_{i=1}^{n} (i+1)(i+2) = \sum_{i=1}^{n} (i^2+3i+2) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{3n(n+1)}{2} + 2n$

$= \dfrac{n(n+1)}{6}[(2n+1)+9] + 2n = \dfrac{n(n+1)}{3}(n+5) + 2n$

$= \dfrac{n}{3}[(n+1)(n+5)+6] = \dfrac{n}{3}(n^2+6n+11)$

34. $\sum_{i=1}^{n} i(i+1)(i+2) = \sum_{i=1}^{n} (i^3+3i^2+2i) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i$

$= \left[\dfrac{n(n+1)}{2}\right]^2 + \dfrac{3n(n+1)(2n+1)}{6} + \dfrac{2n(n+1)}{2}$

$= n(n+1)\left[\dfrac{n(n+1)}{4} + \dfrac{2n+1}{2} + 1\right] = \dfrac{n(n+1)}{4}(n^2+n+4n+2+4)$

$= \dfrac{n(n+1)}{4}(n^2+5n+6) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

35. $\sum_{i=1}^{n} (i^3-i-2) = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 = \left[\dfrac{n(n+1)}{2}\right]^2 - \dfrac{n(n+1)}{2} - 2n$

$= \frac{1}{4}n(n+1)[n(n+1)-2] - 2n = \frac{1}{4}n(n+1)(n+2)(n-1) - 2n$

$= \frac{1}{4}n[(n+1)(n-1)(n+2) - 8] = \frac{1}{4}n[(n^2-1)(n+2)-8] = \frac{1}{4}n(n^3+2n^2-n-10)$

36. By Theorem 3(c) we have that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2} = 78 \quad \Leftrightarrow \quad n(n+1) = 156 \quad \Leftrightarrow \quad n^2+n-156 = 0 \quad \Leftrightarrow$

$(n+13)(n-12) = 0 \quad \Leftrightarrow \quad n = 12$ or -13. But $n = -13$ produces a negative answer for the sum, so $n = 12$.

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37. By Theorem 2(a) and Example 3, $\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = cn$.

38. Let S_n be the statement that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

1. S_1 is true because $1^3 = \left(\frac{1\cdot 2}{2}\right)^2$.

2. Assume S_k is true. Then $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$, so
$$\sum_{i=1}^{k+1} i^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 = \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}(k+2)^2 = \left(\frac{(k+1)[(k+1)+1]}{2}\right)^2$$
showing that S_{k+1} is true.

Therefore, S_n is true for all n by mathematical induction.

39. $\sum_{i=1}^{n}[(i+1)^4 - i^4] = (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \cdots + \left[(n+1)^4 - n^4\right]$
$$= (n+1)^4 - 1^4 = n^4 + 4n^3 + 6n^2 + 4n$$

On the other hand,
$$\sum_{i=1}^{n}[(i+1)^4 - i^4] = \sum_{i=1}^{n}(4i^3 + 6i^2 + 4i + 1) = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 4S + n(n+1)(2n+1) + 2n(n+1) + n \qquad \left[\text{where } S = \sum_{i=1}^{n} i^3\right]$$
$$= 4S + 2n^3 + 3n^2 + n + 2n^2 + 2n + n = 4S + 2n^3 + 5n^2 + 4n$$

Thus, $n^4 + 4n^3 + 6n^2 + 4n = 4S + 2n^3 + 5n^2 + 4n$, from which it follows that

$4S = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$ and $S = \left[\frac{n(n+1)}{2}\right]^2$.

40. The area of G_i is
$$\left(\sum_{k=1}^{i} k\right)^2 - \left(\sum_{k=1}^{i-1} k\right)^2 = \left[\frac{i(i+1)}{2}\right]^2 - \left[\frac{(i-1)i}{2}\right]^2 = \frac{i^2}{4}\left[(i+1)^2 - (i-1)^2\right]$$
$$= \frac{i^2}{4}\left[(i^2 + 2i + 1) - (i^2 - 2i + 1)\right] = \frac{i^2}{4}(4i) = i^3$$

Thus, the area of $ABCD$ is $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

41. (a) $\sum_{i=1}^{n}\left[i^4 - (i-1)^4\right] = \left(1^4 - 0^4\right) + \left(2^4 - 1^4\right) + \left(3^4 - 2^4\right) + \cdots + \left[n^4 - (n-1)^4\right] = n^4 - 0 = n^4$

(b) $\sum_{i=1}^{100}\left(5^i - 5^{i-1}\right) = \left(5^1 - 5^0\right) + \left(5^2 - 5^1\right) + \left(5^3 - 5^2\right) + \cdots + \left(5^{100} - 5^{99}\right) = 5^{100} - 5^0 = 5^{100} - 1$

(c) $\sum_{i=3}^{99}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = \frac{1}{3} - \frac{1}{100} = \frac{97}{300}$

(d) $\sum_{i=1}^{n}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) = a_n - a_0$

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42. Summing the inequalities $-|a_i| \le a_i \le |a_i|$ for $i = 1, 2, \dots, n$, we get $-\sum_{i=1}^{n} |a_i| \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} |a_i|$. Since $|x| \le c \quad \Leftrightarrow$ $-c \le x \le c$, we have $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$. *Another method:* Use mathematical induction.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{i^3}{n^4} + \frac{1}{n}\right] = \lim_{n\to\infty} \left[\frac{1}{n^4} \sum_{i=1}^{n} i^3 + \frac{1}{n} \sum_{i=1}^{n} 1\right] = \lim_{n\to\infty} \left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{n}(n)\right]$

$$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^2 + 1 = \frac{1}{4} + 1 = \frac{5}{4}$$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{16}{n^4} i^3 + \frac{20}{n^2} i\right] = \lim_{n\to\infty} \left[\frac{16}{n^4} \sum_{i=1}^{n} i^3 + \frac{20}{n^2} \sum_{i=1}^{n} i\right]$

$$= \lim_{n\to\infty} \left[\frac{16}{n^4} \frac{n^2(n+1)^2}{4} + \frac{20}{n^2} \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[\frac{4(n+1)^2}{n^2} + \frac{10n(n+1)}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[4\left(1+\frac{1}{n}\right)^2 + 10\left(1+\frac{1}{n}\right)\right] = 4 \cdot 1 + 10 \cdot 1 = 14$$

46. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3 - 2\left(1+\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3} - 2 - \frac{6i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{81}{n^4} i^3 + \frac{81}{n^3} i^2 + \frac{9}{n^2} i - \frac{3}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \frac{n^2(n+1)^2}{4} + \frac{81}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{9}{n^2} \frac{n(n+1)}{2} - \frac{3}{n} n\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 + \frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{9}{2}\left(1+\frac{1}{n}\right) - 3\right]$$

$$= \frac{81}{4} + \frac{54}{2} + \frac{9}{2} - 3 = \frac{195}{4}$$

47. Let $S = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplying both sides by r gives us $rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$. Subtracting the first equation from the second, we find $(r-1)S = ar^n - a = a(r^n - 1)$, so $S = \dfrac{a(r^n-1)}{r-1}$ [since $r \ne 1$].

48. $\sum_{i=1}^{n} \frac{3}{2^{i-1}} = 3\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i-1} = \dfrac{3\left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1}$ [using Exercise 47 with $a = 3$ and $r = \frac{1}{2}$] $= 6\left[1 - \left(\frac{1}{2}\right)^n\right]$

49. $\sum_{i=1}^{n} (2i + 2^i) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \cdot 2^{i-1} = 2\dfrac{n(n+1)}{2} + \dfrac{2(2^n-1)}{2-1} = 2^{n+1} + n^2 + n - 2.$

For the first sum we have used Theorems 2(a) and 3(c), and for the second, Exercise 47 with $a = r = 2$.

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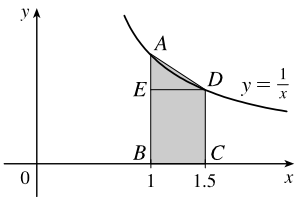
50. $\displaystyle\sum_{i=1}^{m}\left[\sum_{j=1}^{n}(i+j)\right]=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}i+\sum_{j=1}^{n}j\right]$ [Theorem 2(b)] $\displaystyle=\sum_{i=1}^{m}\left[ni+\frac{n(n+1)}{2}\right]$ [Theorem 3(b) and 3(c)]

$$=\sum_{i=1}^{m}ni+\sum_{i=1}^{m}\frac{n(n+1)}{2}=\frac{nm(m+1)}{2}+\frac{nm(n+1)}{2}=\frac{nm}{2}\,(m+n+2)$$

G The Logarithm Defined as an Integral

1. (a)



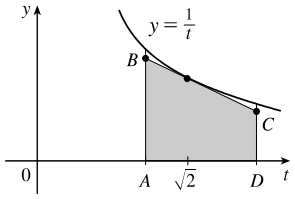
We interpret $\ln 1.5$ as the area under the curve $y=1/x$ from $x=1$ to $x=1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2}\cdot\frac{1}{2}\left(1+\frac{2}{3}\right)=\frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3}<\ln 1.5<\frac{5}{12}$.

(b) $\ln x=\int_1^x(1/t)\,dt$, so $\ln 1.5=\int_1^{1.5}(1/t)\,dt$. With $f(t)=1/t$, $n=10$, and $\Delta t=\frac{1.5-1}{10}=0.05$, we have

$$\ln 1.5=\int_1^{1.5}(1/t)\,dt\approx(0.05)[f(1.025)+f(1.075)+\cdots+f(1.475)]=(0.05)\left[\tfrac{1}{1.025}+\tfrac{1}{1.075}+\cdots+\tfrac{1}{1.475}\right]$$
$$\approx 0.4054$$

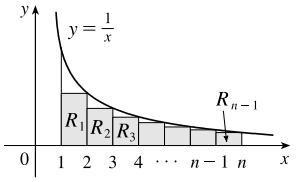
2. (a) $y=\dfrac{1}{t}$, $y'=-\dfrac{1}{t^2}$. The slope of the line through $A(1,1)$ and $D\left(2,\frac{1}{2}\right)$ is $\dfrac{1/2-1}{2-1}=-\dfrac{1}{2}$. Let c be the t-coordinate of the point on $y=\dfrac{1}{t}$ with slope $-\dfrac{1}{2}$. Then $-\dfrac{1}{c^2}=-\dfrac{1}{2}\ \Rightarrow\ c^2=2\ \Rightarrow\ c=\sqrt{2}$ since $c>0$. Therefore, the tangent line is given by $y-\frac{1}{\sqrt{2}}=-\frac{1}{2}\left(t-\sqrt{2}\right)$, or $y=-\frac{1}{2}t+\sqrt{2}$.

(b)

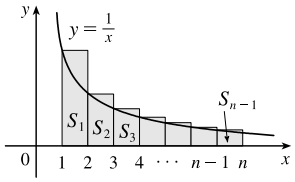


Since the graph of $y=1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC. Now $|AB|=-\frac{1}{2}+\sqrt{2}$ and $|CD|=-1+\sqrt{2}$. The area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2}+\sqrt{2}\right)+\left(-1+\sqrt{2}\right)\right]1=-\frac{3}{4}+\sqrt{2}\approx 0.6642$. So $\ln 2>$ area of trapezoid $ABCD>0.66$.

3.



The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\displaystyle\int_1^n\frac{1}{t}\,dt=\ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1+\dfrac{1}{2}+\cdots+\dfrac{1}{n-1}>\displaystyle\int_1^n\frac{1}{t}\,dt=\ln n$.

Thus, $\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\cdots+\dfrac{1}{n-1}$.

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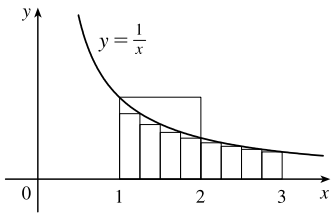
4. (a) From the diagram, we see that the area under the graph of $y = 1/x$ between $x = 1$ and $x = 2$ is less than the area of the square, which is 1. So $\ln 2 = \int_1^2 (1/x)\,dx < 1$. To show the other side of the inequality, we must find an area larger than 1 which lies under the graph of $y = 1/x$ between $x = 1$ and $x = 3$. One way to do this is to partition the interval $[1, 3]$ into 8 intervals of equal length and calculate the resulting Riemann sum, using the right endpoints:



$$\frac{1}{4}\left(\frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4} + \frac{1}{2} + \frac{1}{9/4} + \frac{1}{5/2} + \frac{1}{11/4} + \frac{1}{3}\right) = \frac{28{,}271}{27{,}720} > 1$$

and therefore $1 < \int_1^3 (1/x)\,dx = \ln 3$.

A slightly easier method uses the fact that since $y = 1/x$ is concave upward, it lies above all its tangent lines. Drawing two such tangent lines at the points $\left(\frac{3}{2}, \frac{2}{3}\right)$ and $\left(\frac{5}{2}, \frac{2}{5}\right)$, we see that the area under the curve from $x = 1$ to $x = 3$ is more than the sum of the areas of the two trapezoids, that is, $\frac{2}{3} + \frac{2}{5} = \frac{16}{15}$. Thus, $1 < \frac{16}{15} < \int_1^3 (1/x)\,dx = \ln 3$.

(b) By part (a), $\ln 2 < 1 < \ln 3$. But e is defined such that $\ln e = 1$, and because the natural logarithm function is increasing, we have $\ln 2 < \ln e < \ln 3 \quad \Leftrightarrow \quad 2 < e < 3$.

5. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)\,(rx^{r-1}) = r/x$. But if $g(x) = r \ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r \ln x + C$. Put $x = 1$: $\ln(1^r) = r \ln 1 + C \quad \Rightarrow \quad C = 0$, so $\ln(x^r) = r \ln x$.

6. Using the second law of logarithms and Equation 10, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since ln is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

7. Using the third law of logarithms and Equation 10, we have $\ln e^{rx} = rx = r \ln e^x = \ln(e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

8. Using Definition 13 and the second law of exponents for e^x, we have $a^{x-y} = e^{(x-y)\ln a} = e^{x \ln a - y \ln a} = \dfrac{e^{x \ln a}}{e^{y \ln a}} = \dfrac{a^x}{a^y}$.

9. Using Definition 13, the first law of logarithms, and the first law of exponents for e^x, we have $(ab)^x = e^{x \ln(ab)} = e^{x(\ln a + \ln b)} = e^{x \ln a + x \ln b} = e^{x \ln a} e^{x \ln b} = a^x b^x$.

10. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \quad \Rightarrow \quad \log_a(xy) = r + s = \log_a x + \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^r}{a^s} = a^{r-s} \quad \Rightarrow \quad \log_a \dfrac{x}{y} = r - s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \quad \Rightarrow \quad \log_a(x^y) = ry = y \log_a x$

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